DYNAMICS OF CERTAIN ENTIRE FUNCTIONS ARISING FROM SEPARATELY CONVERGENT CONTINUED FRACTIONS

by

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DEPARTMENT OF MATHEMATICS

Indian Institute of Technology Kanpur

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DYNAMICS OF CERTAIN ENTIRE FUNCTIONS ARISING FROM SEPARATELY CONVERGENT CONTINUED FRACTIONS

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M. GURU PREM PRASAD

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CERTIFICATE

It is certified that the work contained in the thesis entitled "**Dynamics of certain entire functions arising from separately convergent continued fractions**", by **M. Guru Prem Prasad**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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December, 1997

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There are a few,
Who are very special
And so dear,
Without whose help,
This task would not have been accomplished.
So,
This opportunity happily I take,
To express my deepest gratitude,
And heart felt thanks,
To those loving hearts.

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His conversations so delightful,
And manners much cheerful,
His guidance readily available even at late hours;
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And made a safe anchorage possible.

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Dr. M. C. Bhandari, Dr. U. B. Tewari and Dr. V. Raghavendra.

My ship became wealthier,
As it touched many a shores,
With the gems and jewels
which my friends,
benefactors and professors,
Both home and abroad gifted me,
Their research papers, reprints
And other materials
The most valuable gifts
I can never imagine to receive.
Hence I thank those loving souls
Who by sharing with me their knowledge,
Hath helped to enrich me.

I thank the deep ocean of knowledge,
the I.I.T. Kanpur
with its powerful waves of financial assistance,
Hath all the while made my ship float safely.

Whenever I had discussions
at the residence of Dr. G. P. Kapoor,
The hospitality of Mrs. G. P. Kapoor
added flavour and fragrance to it.
Hence
I thank the couple for their warm treatment and hospitality.

Though staying far away from home
I was always made to feel at home
A host of my friends
With their pleasant manners,
Pleasing attitude and timely help
Made my stay here a memorable one.
I thank them all
For the cherishable moments
I had enjoyed here.

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M. Guru Prem Prasad

DEDICATED

TO

THE LOVING MEMORY OF MY PARENTS

Shri MAHALINGAM & Smt. VALLIAMMAL

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Chapter 1

Introduction

Complex analytic dynamics is one of the most alluring area in the field of dynamical systems, since analytic, non-invertible dynamical systems of the Riemann sphere are surprisingly intricate and beautiful. Especially, the study of chaotic systems has been hailed as one of the important breakthroughs in science in this century. The field of complex analytic dynamics traces its origin to the late nineteenth and early twentieth century. At that time, mathematicians such as Böttcher, Koenigs, Leau, Schröder, among others, became interested in the local behavior of complex functions under iteration near a fixed point. During the period 1918-20, the French mathematicians Gaston Julia and Pierre Fatou took a global point of view instead of considering only local dynamical behavior. The pioneering work of Fatou and Julia in the early twentieth century gave a sound footing to the study of complex analytic dynamics. After a short period of intense study during 1910-1925, the field all but disappeared. Since the last decade there has been a renewed interest in the complex analytic dynamics, partially due to the beautiful computer graphics related to it and partially, due to new and powerful mathematical techniques generated by it.

In complex analytic dynamics, the iteration theory of rational functions originated in the work of Fatou [41, 42] and Julia [58, 59] in 1918-20. After this, there was not much activity in the field for fifty years. But there were two notable events during this period of sluggish growth. In 1942, Siegel [89] showed that Siegel disks could in fact

exist in complex dynamical systems, which Fatou and Julia anticipated to occur. Later, Baker [2–12] extended much of the work of Fatou and Julia to the class of entire functions, showing along the way that a new type of stable behavior (Wandering domains) could occur for entire transcendental functions.

The second major part of activity began in 1980 after the use of computer graphics in this subject. In 1980, Mandelbrot first used computer graphics to explore complex dynamics. Mandelbrot's discovery of the $\mathcal{M}$ (*Mandelbrot*) set [71–73] inspired many mathematicians to reinvestigate this field. Dennis Sullivan [92–94] introduced the use of quasi-conformal mappings into the subject. Using quasi-conformal mappings, he proved 'No Wandering Domain theorem' for rational functions, which essentially completed the classification of stable dynamics for rational maps. At the same time Douady and Hubbard [38] opened new vistas in the field by considering the parameter space for quadratic polynomials. They developed a technique which enabled them to classify more or less completely all possible types of quadratic dynamics. The iterations of entire and meromorphic functions has been studied mainly by Baker *et al.* [2–16]. As compared to the case of rational functions, much less work has been done to study the iterations of entire transcendental functions in general. Recently, Devaney [26–29, 31–33, 36, 37] Durkin [33], Eremenko [39], Goldberg [48], Keen [48], Krych [36], Lyubich [39] and Tangerman [37] studied exhaustively the dynamics of certain critically finite entire transcendental functions. The dynamics of critically finite transcendental functions like λe^z, $\lambda \cos z$, and $\lambda \sin z$ are extremely interesting although somewhat intricate.

1.1 Basic theory in complex analytic dynamics

In this section, we review some of the basic definitions and results in iteration theory that concern the dynamics of complex analytic functions studied in the sequel.

In complex dynamical systems, we consider only the iteration of complex analytic func-

tions and look at the behaviour of the resulting orbits for various initial points. When Julia and Fatou first studied these orbits, they found that while some orbits exhibited very tame, or stable behaviour, others behaved in a chaotic or unstable manner. In honor of the contributions of these two mathematicians, we now call the stable region for a complex dynamical system the *Fatou set*, while the chaotic region is known as the *Julia set.* The set of all chaotic orbits, the Julia set, has been the subject of extensive contemporary research. It is well known that for many simple dynamical systems, chaotic sets form *fractals* [71, 78, 79]. Fractals have become enormously popular as a model for a wide variety of physical phenomena [51] and as an art form with the advent of computer graphics.

Fatou and Julia sets

In order to give the precise definitions of the Julia set and the Fatou set, the following basic concepts from the theory of normal families are needed:

Consider in $\mathbb{R}^3$ the sphere $S : x_1^2 + x_2^2 + (x_3 - \frac{1}{2})^2 = \frac{1}{4}$. The x_1x_2- plane is tangent to S at $(0,0,0)$. Let the line from $N = (0,0,1)$, called the *North pole*, to a point z_1 in the x_1x_2-plane intersects the sphere at a point P_1. This gives a one-to-one correspondence $z_1 \leftrightarrow P_1$ between the points of complex plane (x_1x_2- plane) $\mathbb{C}$ and $S \setminus \{N\}$. The correspondence is extended to the one between S and $\mathbb{C}^\infty = \mathbb{C} \cup \{\infty\}$ by associating the point N with ∞. The sphere S is called *Riemann sphere*, $\mathbb{C}^\infty$ is called the *extended complex plane*, and the correspondence is called a *stereographic projection.*

Let z_1 and z_2 be the two points in $\mathbb{C}^\infty$ corresponding to the points P_1 and P_2 respectively, on the Riemann sphere S. The *chordal distance* between z_1 and z_2 is defined

for $z_1, z_2 \in \mathbb{C}$,

$$\chi(z_2, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}} ,$$

while, for $z_2 = \infty$,

$$\chi(z_1, \infty) = \frac{1}{\sqrt{1+|z_1|^2}} .$$

Here, $|z_1 - z_2|$ is the distance between z_1 and z_2 in $\mathbb{R}^2$ with respect to the usual (Euclidean) metric. Clearly, $\chi(z_1, z_2) \leq |z_1 - z_2|$, $\chi(z_1, z_2) = \chi(\frac{1}{z_1}, \frac{1}{z_2})$, and $\chi(z_1, z_2) \leq 1$ in $\mathbb{C}^\infty$.

A sequence of functions $\{f_n\}$, $f_n : \mathbb{C}^\infty \to \mathbb{C}^\infty$, $n = 1, 2, \cdots$ converges *spherically uniformly* to a function f on a set $E \subseteq \mathbb{C}^\infty$ if, for any $\varepsilon > 0$, there is a number $n_0 \equiv n_0(\varepsilon)$ such that $n \geq n_0$ implies that, for all $z \in E$, $\chi(f(z), f_n(z)) < \varepsilon$.

Definition 1.1.1. *A family $\mathcal{F}$ of analytic functions defined on a domain $\Omega \subseteq \mathbb{C}$ is said to be* **normal** *in Ω if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ contains either a subsequence which converges (in usual metric) to a limit function f uniformly on each compact subset of Ω, or a subsequence which converges (in usual metric) uniformly to ∞ on each compact subset. The family $\mathcal{F}$ is said to be normal at a point $z_0 \in \Omega$ if it is normal in some neighborhood of z_0.*

It follows that a family of analytic functions $\mathcal{F}$ is normal in a domain Ω if and only if $\mathcal{F}$ is normal at each point of Ω. If f_n converges uniformly to $f \not\equiv \infty$ on each compact subset of Ω, by the Weirstrass theorem, the limit function $f(z)$ is analytic in Ω, while if f_n converges uniformly to ∞ on each compact subset of Ω then, for each compact subset $K \subseteq \Omega$ and any constant M, $|f_n(z)| > M$ for all $z \in K$ and sufficiently large n (possibly depending on K and M).

The concept of normality for a family of meromorphic functions *i.e.*, functions analytic in the whole complex plane $\mathbb{C}$ except for having poles, is described in the following:

Definition 1.1.2. *A family $\mathcal{F}$ of functions meromorphic in a domain $\Omega \subseteq \mathbb{C}^{\infty}$ is said to be* **normal** *in Ω if every sequence of functions $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converges spherically uniformly on compact subsets of Ω.*

The uniform convergence of a sequence of analytic functions on a domain Ω implies the spherical convergence. Whenever the limit function $f(z)$ is analytic, the converse also holds. Further, if $f \equiv \infty$, $\chi(f_n, \infty) = 1/(\sqrt{1+|f_n|^2})$. Thus,

Proposition 1.1.1. *A family $\mathcal{F}$ of analytic functions is normal in a domain $\Omega \subseteq \mathbb{C}$ with respect to the usual metric (Euclidean metric) if and only if $\mathcal{F}$ is normal in Ω with respect to the chordal metric.*

Montél gave the following Fundamental Normality Test:

Theorem 1.1.1 ([76]). *Let $\mathcal{F}$ be the family of analytic functions on a domain $\Omega \subseteq \mathbb{C}$ that omits two distinct values a and b in $\mathbb{C}$. Then, $\mathcal{F}$ is a normal family in Ω.*

The Fundamental Normality Test has the following analogue for meromorphic functions:

Theorem 1.1.2. *Let $\mathcal{F}$ be the family of meromorphic functions on a domain $\Omega \subseteq \mathbb{C}^{\infty}$ that omits three distinct values a, b, c in $\mathbb{C}$. Then $\mathcal{F}$ is a normal family in Ω.*

Let $f(z)$ be either a non-constant entire function (*i.e.*, analytic in the whole complex plane $\mathbb{C}$) or a rational function of degree greater than one (*i.e.*, quotient of two polynomials with degree of atleast one polynomial being greater than one). Let $z_0 = f^0(z_0)$ and $z_n = f(z_{n-1}) = f^n(z_0); n = 1, 2, \ldots$, where $f^n = f \circ f \circ f \circ \cdots \circ f (n\ times)$ is the nth iterate of f. The points z_n, $n = 0, 1, 2, \cdots$ are called *iterates* of z_0 under the function f. The set $O^+(z_0) = \{f^n(z_0);\ n = 0, 1, 2, \ldots\}$ is called the *orbit* or *forward orbit* of z_0 and the set $O^-(z_0) = \{z : f^n(z) = z_0;$ for some positive integer $n\}$ is called the *backward orbit* of z_0. The complex dynamics problem is to understand the fate of all the points in $\mathbb{C}$ under iterations of the function $f(z)$.

Definition 1.1.3. *Let* $\mathbb{C}^\infty$ *denote the extended complex plane. The* **Fatou set of an entire function** $f(z)$, *denoted by* $\mathcal{F}(f)$, *is defined as*

$$\mathcal{F}(f) = \{z \in \mathbb{C}^\infty : \{f^n\} \text{ is normal at } z\}$$

Definition 1.1.4. *The* **Julia set of an entire function** $f(z)$, *denoted by* $\mathcal{J}(f)$, *is defined as the complement in the extended complex plane* $\mathbb{C}^\infty$ *of the Fatou set of* $f(z)$. *Thus,*

$$\mathcal{J}(f) = \{z \in \mathbb{C}^\infty : \{f^n\} \text{ is not normal in any neighborhood of } z\}$$

For the entire function $f(z) = z^2$, $f^n(z) \to 0$ as $n \to \infty$ for $z \in U \equiv \{z \in \mathbb{C} : |z| < 1\}$ and $f^n(z) \to \infty$ as $n \to \infty$ for $z \in V \equiv \{z \in \mathbb{C} : |z| > 1\}$. Therefore, the Fatou set $\mathcal{F}(f)$ of the function $f(z)$ is $\{z \in \mathbb{C}^\infty : |z| \neq 1\}$ and the Julia set $\mathcal{J}(f)$ is $\{z \in \mathbb{C} : |z| = 1\}$.

For a meromorphic function, the Fatou set and the Julia set are defined as follows:

Definition 1.1.5. *Let* $f(z)$ *be a meromorphic function in* $\mathbb{C}$. *The* **Fatou set** *of* $f(z)$, *denoted by* $\mathcal{F}(f)$, *is defined as*

$$\mathcal{F}(f) = \{z \in \mathbb{C}^\infty : \text{Each } f^n,\ n = 0, 1, 2, \cdots, \text{ is defined and } \{f^n\} \text{ is normal at } z\}$$

where, $\mathbb{C}^\infty$ *denote the extended complex plane. The* **Julia set** *of* $f(z)$, *denoted by* $\mathcal{J}(f)$, *is defined as* $\mathcal{J}(f) = \mathbb{C}^\infty \setminus \mathcal{F}(f)$.

It follows by either of the Definitions 1.1.3 and 1.1.5 that the Fatou set $\mathcal{F}(f)$ is open and consequently, the Julia set $\mathcal{J}(f)$ is closed. The following are some of the elementary properties of the Fatou set and the Julia set of $f(z)$ (see *e.g.*, [17,19,22,65]):

Proposition 1.1.2. *Let* $f(z)$ *be either an entire function or a rational function. Then,* $\mathcal{F}(f) = \mathcal{F}(f^n)$ *and* $\mathcal{J}(f) = \mathcal{J}(f^n)$ *for all* $n \geq 2$.

Proposition 1.1.3. *Let $f(z)$ be either an entire function or a rational function. Then, either $\mathcal{J}(f) = \mathbb{C}^{\infty}$ or $\mathcal{J}(f)$ has empty interior.*

Proposition 1.1.4. *Let $f(z)$ be either an entire function or a rational function. Then, $\mathcal{J}(f)$ is non-empty perfect set.*

Definition 1.1.6. *A set S is called* **invariant** *under $z \to f(z)$ if $f(S) \subseteq S$. The set S is called* **completely invariant** *if $f(S) \subseteq S$ and $f^{-1}(S) \subseteq S$.*

Proposition 1.1.5. *Let $f(z)$ be either an entire function or a rational function. Then, $\mathcal{F}(f)$ and $\mathcal{J}(f)$ are completely invariant.*

Proposition 1.1.6. *Let $f(z)$ be either an entire function or a rational function. If $z_0 \in \mathcal{J}(f)$ is not a exceptional value (i.e., $f(z) = z_0$ for some $z \in \mathbb{C}$), then $\mathcal{J}(f) = \overline{O^{-}(z_0)}$.*

Periodic points

The orbits of certain points can be quite complicated sets, even for very simple non-linear mappings. However, there are some orbits which are especially simple and play a central role in the study of the dynamical systems. Such orbits are known as *periodic orbits*. The following is a brief review of the definitions and results concerning periodic points and periodic orbits:

Definition 1.1.7. *A point z is said to be a* **periodic point** *of period p for a function $f(z)$ if $f^p(z) = z$. The least positive integer p for which $f^p(z) = z$ is called the* **minimal period** *of z. The number $\lambda = (f^p)'(z)$ is called the* **multiplier** *or* **eigen value** *of the periodic point z.*

If the minimal period of z is 1 (*i.e.*, $f(z) = z$) then z is called a *fixed point* of $f(z)$. For a periodic point z_0 of period p, the orbit

$$\left\{z_0, z_1 = f(z_0), z_2 = f^2(z_0), \ldots, z_{p-1} = f^{p-1}(z_0)\right\}$$

is called a *cycle* or a *periodic cycle* of z_0.

The periodic point z_0 of period p is classified according to the magnitude of its multiplier $\lambda = (f^p)'(z_0)$ as follows:

Classification of periodic points

- If $|\lambda| < 1$ then the periodic point z_0 is called *attracting.*
- If $|\lambda| > 1$ then the periodic point z_0 is called *repelling.*
- If $|\lambda| = 1$ then the periodic point z_0 is called *neutral* or *indifferent.*

If $\lambda = 0$, the attracting periodic point z_0 is called *superattracting.* If $|\lambda| \neq 1$, the periodic point z_0 is called *hyperbolic.* When $\lambda = e^{2\pi i\alpha}$ the indifferent periodic point is further classified as *rationally indifferent* or *irrationally indifferent* according as α is a rational or an irrational number.

The cycle of the periodic point z_0 is classified as attracting, repelling, rationally neutral or irrationally neutral according as z_0 is attracting, repelling, rationally neutral or irrationally neutral respectively.

It is easily seen that a rational function always has a fixed point. However, a transcendental entire function need not have any fixed point *e.g.*, consider $f(z) = e^z + z$. Fatou [43] proved that a rational function $f(z)$ (of degree > 1) has periodic point of (not necessarily minimal) period n for all $n \geq 1$, while an entire transcendental function $f(z)$ has atleast one periodic point of period 2. The latter result is further generalized by Rosenbloom as follows:

Theorem 1.1.3 ([81]). *An entire transcendental function has infinitely many periodic points of period n (not necessarily minimal period) for all $n \geq 2$.*

The following result, conjectured by Baker [7] in 1968, was proved by Bergweiler in 1991:

Theorem 1.1.4 ([18]). *Let $f(z)$ be an entire transcendental function and $n \geq 2$. Then, f has infinitely many periodic points of minimal period n.*

The importance of the periodic points is illustrated by the following theorem of Baker giving a characterization of the Julia set of an entire transcendental function $f(z)$:

Theorem 1.1.5 ([7]). *If $f(z)$ is an entire transcendental function, the Julia set $\mathcal{J}(f)$ is the closure of the set of repelling periodic points of $f(z)$.*

For a rational function $f(z)$, the above result was obtained by both Fatou [42] and Julia [59] independently.

It is easily seen that the attracting periodic points are in the Fatou set, while repelling periodic points are in the Julia set. Further, it is well known that rationally indifferent periodic points are in the Julia set. However, the irrationally indifferent periodic points, may lie either in the Fatou set or the Julia set. Both possibilities do occur [18].

Definition 1.1.8. *The* **basin of attraction** *(or* **attractive basin***) of the attracting periodic point z_0 of a function $f(z)$, denoted by $\mathsf{A}(z_0)$, is the set of points whose orbits tend to z_0 under iteration. i.e., $\mathsf{A}(z_0) = \{z : f^n(z) \to z_0 \text{ as } n \to \infty\}$. The connected component of $\mathsf{A}(z_0)$ containing the periodic point z_0 is called the* **immediate basin of attraction** *(or* **immediate attractive basin***) of z_0 and is denoted by $\mathsf{A}^*(z_0)$.*

The components of Fatou set

Let $f(z)$ be either an entire function or a rational function. A maximal connected domain U contained in the Fatou set of a function $f(z)$ is said to be a component of the Fatou set $\mathcal{F}(f)$. Since $\mathcal{F}(f)$ is completely invariant (c.f. Proposition 1.1.5) and $f^n(z)$ is analytic in U for each n, $f^n(U) = U_n$ *(say)* is a component contained in $\mathcal{F}(f)$

Definition 1.1.9. *A component U of the Fatou set $\mathcal{F}(f)$ is called* **periodic** *with* **period** *n if $f^n(U) = U$. The set $\{U_0 = U, f(U), f^2(U), \ldots, f^{n-1}(U)\}$ is called the (periodic)* **cycle**

of components. The least positive integer n with this property is called the **minimal period** *of U.*

Definition 1.1.10. *A component U of the Fatou set $\mathcal{F}(f)$ is called* **preperiodic** *if there exist non-negative integers n and m with $n > m \geq 0$ such that $f^n(U) = f^m(U)$.*

It is easily seen that the periodic components are preperiodic and if U is preperiodic component then $f^m(U)$ is a periodic component of period $n - m$.

Definition 1.1.11. *A component of $\mathcal{F}(f)$ which is not preperiodic is called a* **wandering component** *or a* **wandering domain.**

Theorem 1.1.6 ([92]). *Let $f(z)$ be a rational function of degree greater than one. Then, the function $f(z)$ has no wandering domain in its Fatou set.*

Sullivan's classification of periodic components

The Fatou set often contains preperiodic components in addition to the periodic one [22]. Sullivan [92–94] completed a classification scheme studied in parts by Fatou, Julia. Siegel, Arnold, Moser and Herman for the dynamics of a rational function in the periodic components of its Fatou set. A similar classification scheme for the class of entire functions is given by Bergweiler. Thus, we have the following theorem giving the behaviour of iterates of an entire function $f(z)$ in the periodic components:

Theorem 1.1.7 ([19]). *Let $f(z)$ be an entire function other than linear polynomial. Let U be a periodic component of the Fatou set $\mathcal{F}(f)$ having minimal period n and let $S = f^n$. Then, only the following possibilities can occur:*

1. *U is an* **attracting domain:** *In this case the periodic component U contains an attracting periodic point z_0 of period n.*

   An attracting domain is also called an **attractive basin** *of z_0. Further, $|S'(z_0)| < 1$, $S^k(z) \to z_0$ for $z \in U$ as $k \to \infty$. The cycle $\{z_0, f(z_0), f^2(z_0), \ldots, f^{n-1}(z_0)\}$ is called*

the **attracting cycle** *for* $f(z)$. *If* $S'(z_0) = 0$, *then* U *is called a* **super attracting domain.**

2. *U is a* **parabolic domain:** *In this case the boundary ∂U of the periodic component U contains a periodic point z_0 of period n and $S^k(z) \to z_0$ for $z \in U$ as $k \to \infty$.*

   A parabolic domain is also called a **Leau domain**. *Further,* $S'(z_0) = 1$. *The cycle* $\{z_0, f(z_0), f^2(z_0), \ldots, f^{n-1}(z_0)\}$ *is called the* **parabolic cycle** *for* $f(z)$.

3. *U is a* **Siegel disk:** *In this case there exists an analytic homeomorphism $\phi : U \to D$ where $D = \{z : |z| < 1\}$ is the unit disk, such that $\phi(S(\phi^{-1}(z))) = e^{2\pi i\,\alpha} z$ for some irrational number α.*

4. *U is a* **Baker domain:** *In this case there exists $z_0 \in \partial U$ such that $S^k(z) \to z_0$ for $z \in U$ as $k \to \infty$, but $S(z_0)$ is not defined. If $f(z)$ is an entire transcendental function, $z_0 = \infty$. Thus, for an entire transcendental function U is also called a* **domain at infinity**. *However, for a polynomial entire function $P(\infty) = \infty$, and hence Baker domains do not exist for polynomials.*

Remark 1.1.1. *A periodic component called* Herman ring *that is different from all the periodic components of Theorem* 1.1.7 *occurs only in the dynamics of rational functions. The component U of the Fatou set of a rational function is called a* **Herman ring** *if there exists an analytic homeomorphism $\phi : U \to A$, A being the annulus $A = \{z : 1 < |z| < r\}$, such that $\phi(S(\phi^{-1}(z))) = e^{2\pi i\,\alpha} z$ for some irrational number α. Herman rings do not exist in the dynamics of entire functions ([101], p. 67).*

The following examples illustrate each of the classifications of periodic components of the Fatou set $\mathcal{F}(f)$ given by Theorem 1.1.7:

Example 1.1.1. Attracting domain

Let $f(z) = z^2$ be an entire function. The point $z = 0$ is an (super) attracting fixed point for $f(z)$. In this case, $U = A(0) = \{z : |z| < 1\}$ is an attracting domain for $f(z)$.

Example 1.1.2. Parabolic domain

Let $E(z) = e^{z-1}$. The point $z = 1$ is a rationally indifferent fixed point of $E(z)$. Let $U =$ interior $\{z \in \mathbb{C} : E^n(z) \to 1$ as $n \to \infty\}$. The fixed point $z = 1$ lies on the boundary of U, since $E^n(x) \to \infty$ for $x > 1$ as $n \to \infty$. Thus, U is a parabolic domain for $E(z)$.

Example 1.1.3. Siegel disk

Let $P(z) = e^{2\pi i\alpha}z + \cdots + z^d$, where α is an irrational number satisfying the condition $\sum_{n=1}^{\infty} (\log(q_{n+1})/q_n) < \infty$ where p_n/q_n, $n = 1, 2, \ldots$ are the continued fraction approximants to α. The point $z = 0$ is irrationally indifferent fixed point of $P(z)$ with multiplier $e^{2\pi i\alpha}$. Further, there exists an analytic homeomorphism $\phi : U \to D$ where D is the unit disk and U is an open neighborhood containing 0 such that $\phi(f(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ [65]. Thus, U is a Siegel disk for $P(z)$.

Example 1.1.4. Baker domain

It is easily seen that a polynomial has no Baker domain, since ∞ is a super attracting fixed point for a polynomial. Let $f(z) = 1 + z + e^{-z}$ be an entire transcendental function. Set $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$ and $U = \{z \in \mathbb{C} : f^n(z) \to \infty$ as $n \to \infty\}$. It is observed that $\Re(f(z)) = 1 + \Re(z) + \Re(e^{-z}) > \Re(z)$ for $z \in H^+$. Therefore, all the orbits in the half plane H^+ lie in stable domain U consisting of points whose orbits tend to the essential singularity ∞. Consequently, $H^+ \subseteq U$ is a Baker domain for $f(z)$.

It is well known that polynomials and rational functions have no wandering domains [39, 48, 92]. However, besides the periodic domains, an entire transcendental function may admit a *wandering domain.*

Example 1.1.5. Wandering domain

Let $f(z) = z + \lambda \sin z$, where $\lambda > 1$ is chosen such that each critical point z_0 of $f(z)$ is

mapped to $z_0 \pm 2\pi$, another critical point of $f(z)$. Thus, there are only two distinct orbits corresponding to the critical points. Consequently, all sufficiently small neighborhoods of a critical point lie in the Fatou set, since the iterates contracts these regions and all orbits tend uniformly to ∞ in these neighborhoods. Further, the vertical lines $x = k\pi$ lie in the Julia set of $f(z)$ for any integer k [29]. It therefore follows that each critical point on a critical orbit lies in a distinct component of the Fatou set. Thus, each of these components are wandering domains since they are not preperiodic.

Singular values

Besides periodic points, there are other points which play equally important role in the dynamics of a function. The following is a review of the role of such points in the complex dynamics.

A point w is said to be a *critical point* of $f(z)$ if $f'(w) = 0$. The value $f(w)$ corresponding to a critical point w is called a *critical value* of f. A point $w \in \mathbb{C}^{\infty}$ is said to be an *asymptotic value* for $f(z)$, if there is a continuous curve $\gamma(t)$ satisfying

$$\lim_{t\to\infty} \gamma(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} f(\gamma(t)) = w \ .$$

Clearly, ∞ is an asymptotic value for every entire function. If a function $f(z)$ has an asymptotic value w, the preimage of any neighborhood of w is unbounded and has noncompact closure.

Definition 1.1.12. *The set $SV(f)$ of* **singular values** *of an entire function $f(z)$ is defined as the union of the set of all critical values of $f(z)$ and the set of all finite asymptotic values of $f(z)$. Thus,*

$$SV(f) = CV(f) \cup AV(f)$$

where, $CV(f)$ = Set of all critical values of $f(z)$ and $AV(f)$ = Set of all (finite) asymptotic values of $f(z)$.

Definition 1.1.13. *An entire function is said to be* **critically finite** *if it has only finitely many asymptotic and critical values. If an entire function $f(z)$ is not critically finite then it is said to be* **non-critically finite.**

The following results exhibit the importance of singular values in the dynamics of an entire function.

Theorem 1.1.8 ([32]). *Let $f(z)$ be an entire function other than a linear polynomial. Suppose z_0 lies on an attracting cycle or a parabolic cycle (c.f. Theorem 1.1.7) of $f(z)$. Then, the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of z_0.*

Theorem 1.1.9 ([32]). *Let $f(z)$ be an entire function other than a linear polynomial and the Fatou set $\mathcal{F}(f)$ has a Siegel disk. Then, the forward orbit of some critical point must accumulate in its boundary.*

In case of domains which are not preperiodic, the finite limit functions of iterates of an entire functions in wandering domains are limit points of the forward orbits of the singular values. More precisely, we have

Theorem 1.1.10 ([20]). *Let $f(z)$ be an entire function and U be a wandering domain of $f(z)$. Let $S'(f) = Der\{f^n(z) : z \in SV(f),\, n = 0, 1, 2, \dots\}$ denote the derived set of the forward orbits of all singular values of f. Then, all limit functions of the sequence $\{f^n|_U\}$ are contained in $S'(f) \cup \{\infty\}$.*

Dynamics of critically finite entire functions

The critically finite entire maps form a class of entire functions whose dynamics prove to be most tractable. We review below some of the recent works on the dynamics of the class of critically finite transcendental entire functions.

Let

$$\mathcal{D} \equiv \{f(z) : f(z) \text{ is a critically finite transcendental entire function}\} .$$

It is easily seen that the entire functions λe^z, $\lambda \sin z$ and $\lambda \cos z$ are in class $\mathcal{D}$. On the other hand $\lambda(e^z - 1)/z$, $\lambda \sinh z/z$ and $z + \lambda \sin z$ are not in $\mathcal{D}$, since they have infinitely many critical values. As a dynamical system, a function in the class $\mathcal{D}$ shares many of the properties of polynomial or rational functions. However, there are several significant differences; such as existence of a wandering domain, existence of a Baker domain, existence of an unbounded domain of attraction for a finite attracting periodic point in the Fatou sets of entire transcendental functions while any of these types of domains can not be contained in the Fatou sets of polynomials or rational functions. The dynamics of the functions in $\mathcal{D}$ is mainly studied by Devaney [26, 27, 31], Durkin [33], Eremenko [39], Goldberg [48], Lyubich [39], Keen [48], Krych [36], and Tangerman [37]. An excellent review of almost all the fundamental results on dynamics of critically finite entire functions is due to Devaney [29, 32].

The Julia sets of certain critically finite entire transcendental functions contain Cantor bouquets - a topological structure that is quite different from kinds of Julia sets that occur in the study of rational functions or polynomials. In the following, we briefly describe the relevant definitions and notations for Cantor bouquets:

Let for a positive integer N, $\Sigma_N = \{(s) = (s_0 s_1 s_2 \ldots) : s_j \in \mathsf{Z}, |s_j| \leq N\}$, where Z is the set of all integers. The set Σ_N consists of all infinite sequences of integers less than or equal to N in absolute value. It is well known that, with the product topology, Σ_N is homeomorphic to a Cantor set [91].

Definition 1.1.14. *Define* $\sigma : \Sigma_N \to \Sigma_N$ *by*

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots)$$

The map σ *is called the* **(right) shift map.**

Definition 1.1.15. *Let $f \in \mathcal{D}$. An invariant subset C of the Julia set $\mathcal{J}(f)$ of a function $f(z)$ is called a* **N-bouquet** *if*

(1) There is a homeomorphism $h : \Sigma_N \times [0, \infty) \to C$

(2) $\pi \circ h^{-1} \circ f \circ h(s,t) = \sigma(s)$, where $\pi : \Sigma_N \times [0, \infty) \to \Sigma_N$ is the projection map

(3) $\lim_{t\to\infty} h(s,t) = \infty$

(4) $\lim_{n\to\infty} f^n \circ h(s,t) = \infty$, *for* $t \neq 0$

The invariance of C requires that $E(h(s,0)) = h(\sigma(s),0)$. Hence the set of points $\Lambda = h(s,0)$ is an invariant set on which f is topologically conjugate to the shift. The set Λ is called the *crown* of C. For each s, the curve $h(s,t)$ for $t > 0$ is called the *tail* associated with s.

Definition 1.1.16. *Let C_n be a n-bouquet and suppose $C_n \subseteq C_{n+1}$, $n = 1, 2, \ldots$. Then, the set $\overline{\cup_{n\geq 0} C_n}$ is called a* **Cantor bouquet**.

The Cantor bouquets were first observed for the functions λe^z in [36]. Devaney and Tangerman [37] later showed that the entire functions which are critically finite and meet certain growth conditions have Cantor bouquets in their Julia sets. The following are relevant definitions and results in this direction:

Definition 1.1.17. *Let R_1 and R_2 be Riemann spheres. An analytic function $f : R_1 \to R_2$ is said to be a* **covering map** *if every $w \in R_2$ lies in a disk U such that each connected component of $f^{-1}(U)$ is mapped conformally by f onto U.*

Theorem 1.1.11 ([37]). *Let $f \in \mathcal{D}$. Let the open disk $B_\rho(0)$, centered at the origin and having radius ρ, contain all the critical values and (finite) asymptotic values and Γ be the complement of $B_\rho(0)$. Then,*

(1) Any connected component T of $f^{-1}(\Gamma)$ is a disk whose closure contains ∞.

(2) $f : T \to \Gamma$ is a covering map.

Definition 1.1.18. *Let $f \in \mathcal{D}$ and Γ be as in Theorem* 1.1.11. *A component T of $f^{-1}(\Gamma)$ is called an* **exponential tract**. *In such a region, the function $f(z)$ can be written as $f(z) = exp\ (\phi(z))$ for $z \in T$ and some analytic map ϕ.*

The specific conditions that guarantee that the set of points whose orbits remain inside a given exponential tract T is a Cantor bouquet, are found in [37].

It is assumed in the sequel that T is contained in a sector S and there exists ρ such that $f|_T$ covers the complement of the disk $B_\rho(0)$ and that $T \cap B_\rho(0) = \emptyset$.

Definition 1.1.19. *A ray $\zeta = \zeta(r) = re^{i\hat{\theta}}$ which is disjoint from S is fixed. Let $\gamma_i = \gamma_i(r)$, $i = 0, \pm1, \pm2, \ldots$ denote the pre-images of ζ in T. i.e. $\gamma_i(r) = f^{-1}(\zeta(r))$ for an appropriate branch of f^{-1}. The index i is chosen so that γ_i and γ_{i+1} are adjacent for each i. Let T_i denote the strip bounded by the curve γ_i and γ_{i+1}. Then, the domain T_i is called a* **fundamental domain** *for $f|_T$.*

Example 1.1.6. *Let $E_\lambda(z) = \lambda\ e^z$, $0 < \lambda < 1/e$. Then, it is easily seen that $E_\lambda(z)$ has only one asymptotic value namely '0' and no critical values. The vertical line $x = \log(1/\lambda)$ is mapped by E_λ onto the unit circle. Also, E_λ maps the half plane $T = \{z : \Re(z) > \log(1/\lambda)\}$ onto the exterior of the unit circle. By Theorem* 1.1.11, *the half plane T is an exponential tract for E_λ. Clearly, $T \cap B_1(0) = \emptyset$ and $T \subset S$ where, $S = \left\{z = re^{i\theta} : (-\pi/2) < \theta < (\pi/2), r > 0\right\}$ The ray $\zeta = \zeta(r) = \{x \in \mathsf{R} : x < -1\}$ is disjoint from S and $E^{-1}(\zeta)$ gives the fundamental domains for $E_\lambda|_T$, $0 < \lambda < 1/e$.*

Set, $W_N = \cup_{i=-N}^{N} T_i$ and $\Lambda_N = \left\{z \in W_N : f^k(z) \in W_N, \text{ for all } k \geq 0\right\}$.

Definition 1.1.20. *Let $f \in \mathcal{D}$ and T be an exponential tract. Then, $f|_T$ is said to have an* **asymptotic direction** *θ^* if $\gamma_i(r)$, $i = 0, \pm1, \pm2, \ldots$, (c.f. Definition* 1.1.19*) is C^1-asymptotic to a straight line with direction θ^* for each curve defining the fundamental domains.*

Definition 1.1.21. *Let $f \in \mathcal{D}$. An exponential tract T is said to be a* **hyperbolic exponential tract** *if there exist positive constants R_1, α, C such that if z and $f(z)$ lie in W_N, with $|z| = r > R_1$, then*

(1) $|f(z)| > C\, e^{r^\alpha}$

(2) $|f'(z)| > C\, e^{r^\alpha}$

(3) $|\arg(f'(z))| > C\, e^{-r^\alpha}$.

The constants R_1, α and C may depend on N.

The following theorem due to Devaney and Tangerman shows that the set Λ_N is homeomorphic to a N-bouquet, under sufficient conditions amounting to an assumption on the rate at which the points in the exponential tract of $f(z)$ approach to ∞ along the asymptotic direction of $f \in \mathcal{D}$:

Theorem 1.1.12 ([37]). *Let $f \in \mathcal{D}$. Let T be a hyperbolic exponential tract on which f has asymptotic direction θ^*. Then for each N, Λ_N is a N-bouquet. Consequently, $\mathcal{J}_T(f) = \left\{z : f^k(z) \in T, \text{for all}\, k \geq 0\right\} \subseteq \mathcal{J}(f)$ contains a Cantor bouquet.*

Devaney and Keen [34, 35] proved that the Julia sets of certain meromorphic functions whose Schwarzian derivatives are polynomials also contain Cantor bouquets. It is worth noting that the meromorphic functions whose Schwarzian derivatives are polynomials have only finitely many asymptotic values and no critical values and hence the behaviour of the orbits of singular values is easily traceable.

From Theorem 1.1.12, Devaney and Tangerman deduced the following result giving a characterization of the Julia set that is found to be extremely helpful in computationally generating the Julia sets of critically finite entire transcendental functions:

Theorem 1.1.13 ([37]). *Let $f \in \mathcal{D}$. Let T be a hyperbolic exponential tract on which f has asymptotic direction θ^*. Then,*

$$\mathcal{J}(f) = Clo\{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

where, $Clo(A)$ denotes the closure of a set A.

The 'no wandering domain' theorem due to Sullivan [92] gives that if $f(z)$ is a rational function of degree greater than 1 then $f(z)$ has no wandering domain in its Fatou set. Goldberg and Keen [48] and Eremenko and Lyubich [39] extended Sullivan's result to the entire functions in the class $\mathcal{D}$ as follows:

Theorem 1.1.14 (**[39,48]**). *Let $f \in \mathcal{D}$. Then, $f(z)$ has no wandering domains.*

Theorem 1.1.15 (**[40]**). *Let $f \in \mathcal{D}$. Then, $f(z)$ has no Baker domains.*

It is well known that the Julia set of a polynomial never equals the extended complex plane, since the point at infinity is an attracting fixed point for a polynomial. Fatou [43] in 1926, conjectured that the Julia set of the function e^z equals the extended complex plane. Misiurewicz [75] proved the Fatou's conjecture affirmatively in 1981. The following theorems give criteria for Julia set of an entire function to be the extended complex plane:

Theorem 1.1.16 (**[31]**). *Let $f \in \mathcal{D}$ and the forward orbits of all its singular values tend to ∞ under iteration of f. Then, the Julia set $\mathcal{J}(f)$ of $f(z)$ equals the extended complex plane $\mathbb{C}^\infty$.*

Theorem 1.1.17 (**[31]**). *Let $f \in \mathcal{D}$ and all its singular values are preperiodic (but not periodic). Then, the Julia set $\mathcal{J}(f)$ of $f(z)$ equals the extended complex plane $\mathbb{C}^\infty$.*

Among the critically finite transcendental functions, it is probably the exponential family λe^z that has received most attention. Devaney and coworkers [26,27,31,33,36,37] exhaustively studied the dynamics of entire functions λe^z, $(\lambda > 0)$ and exhibited all its beauties. Some of the main results on the dynamics of the entire transcendental function $E_\lambda(z) = \lambda e^z$, $\lambda > 0$ are reviewed in the following:

Theorem 1.1.18 (**[33]**). *Let $E_\lambda(z) = \lambda e^z$, $\lambda > 0$. Then, the Julia set of $E_\lambda(z)$ is a nowhere dense subset of the right half plane for $0 < \lambda < (1/e)$.*

Theorem 1.1.19 ([33]). *Let $E_\lambda(z) = \lambda e^z$, $\lambda > 0$. Then, the Fatou set of $E_\lambda(z)$, $0 < \lambda < (1/e)$, is the attractive basin $\mathsf{A}(a_\lambda)$ of the attracting real fixed point a_λ of $E_\lambda(z)$.*

Theorem 1.1.20 ([37]). *Let $E_\lambda(z) = \lambda e^z$, $\lambda > 0$. Then, the Julia set of $E_\lambda(z)$ contains Cantor bouquets for $0 < \lambda < (1/e)$.*

Theorem 1.1.21 ([31]). *Let $E_\lambda(z) = \lambda e^z$, $\lambda > 0$. Then, for $\lambda > (1/e)$ Julia set of $E_\lambda(z)$, equals the extended complex plane C^∞.*

Devaney and Durkin [33] while describing the Julia set $\mathcal{J}(E_\lambda)$ of $E_\lambda(z) = \lambda e^z$, $\lambda > 0$, exhibited the interesting phenomena of *explosion* in the Julia sets of functions in one parameter family $\mathcal{E} = \{\lambda e^z \ : \ \lambda > 0\}$. They proved that the Julia set of $E_\lambda(z)$ for $0 < \lambda < (1/e)$ is a nowhere dense subset entirely contained in the right half plane. As soon as the parameter λ crosses the value $(1/e)$, $\mathcal{J}(E_\lambda)$ suddenly explodes and equals to the extended complex plane. This phenomena is referred to as *explosion* or *chaotic burst* in the Julia sets of functions in one parameter family. This type of explosion occurs as well in other family of functions like $\mathcal{C} = \{i\,\lambda\cos z \ : \ \lambda > 0\}$.

The characterization of the Julia set of $E_\lambda(z)$ as the closure of the set of all *escaping points* (*i.e.* the points whose orbits tend to ∞ under iteration) is as follows:

Theorem 1.1.22 ([33,37]). *Let $E_\lambda(z) = \lambda e^z$, $\lambda > 0$. Then, the Julia set $\mathcal{J}(E_\lambda)$ of $E_\lambda(z)$ is given by $\mathcal{J}(E_\lambda) = clo\{z : E_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$*

It is observed that the dynamics of functions belonging to certain one parameter family are unchanged for large intervals of parameter values. At a particular parameter value, the dynamics changes suddenly, after which it again remains the same for the parameter belonging to a large interval. These sudden changes in dynamics are called *bifurcations.* The presence of nonhyperbolic periodic points for parameter values at which there is a bifurcation is quite commonly encountered.

Definition 1.1.22. *Let $f_\lambda(z)$, $\lambda \in \mathrm{R}$ be a parametrized family of functions. A* **bifurcation** *is said to occur at the parameter value λ_0 if there exists $\epsilon > 0$ such that whenever a and b satisfy $\lambda_0 - \epsilon < a < \lambda_0$ and $\lambda_0 < b < \lambda_0 + \epsilon$, the dynamics of $f_a(z)$ are different from the dynamics of $f_b(z)$. In other words, the dynamics of the function changes when the parameter value crosses through the point λ_0.*

A bifurcation occurring in the dynamics of $E_\lambda(z) = \lambda e^z$, $\lambda > 0$ is observed by Devaney. Thus,

Theorem 1.1.23 (**[26,31]**). *Let $\mathcal{E} = \{E_\lambda(z) = \lambda e^z \ : \ \lambda > 0\}$ be a one parameter family of functions. Then, a bifurcation in the dynamics of $E_\lambda(z)$ occurs at the parameter value $\lambda = (1/e)$.*

This type of bifurcation is also found to occur in other families of functions like $\{\lambda \sin z : \lambda > 0\}$ and $\{i\,\lambda \cos z : \lambda > 0\}$.

1.2 Basic theory in continued fractions

The field of continued fractions has been a rich, vast and diverse subject fascinating the mathematicians since the 16th century. The theory of continued fractions originated in its explicit form in the work of Bombelli in 1572. The earliest continued fractions had whole numbers as their entries and were applied to the rational approximation of various algebraic numbers and π. Continued fraction expansions involving functions of a complex variable rather than simply numbers were introduced by Euler and became an important tool in the approximation of special classes of analytic functions in the works of Euler, Lambert and Lagrange. A particularly useful direction of study in which orthogonal polynomials made their appearance was the expansions of ratios of hypergeometric functions in continued fractions, introduced by Gauss in 1813. Ever since this theory has grown significantly with

the contributions of several mathematicians. The development of continued fraction theory with historical references is found in [55]. Other recent developments can be found in [68].

The continued fractions often provide representations for transcendental functions that are much more generally valid than their classical representation by the power series. Thus, for instance, while the power series at $z = 0$ of a meromorphic function represents that function only upto the nearest pole, continued fraction representations exist for certain meromorphic functions which represent that function everywhere in the complex plane except at the poles. Many a times continued fraction of a function may converge faster than the power series expansion. The orthogonal polynomials can be obtained as denominators of the approximants of certain continued fractions.

Classically, the continued fractions have a wide range of applications dealing with analytic continuation, location of zeros and singular points, stable polynomials, acceleration of convergence, summation of divergent series, asymptotic expansions, orthogonal polynomials, moment problems and birth-death processes. In recent years there has been a renewed interest in the subject of continued fractions due in part to the importance of the algorithmic character of continued fractions. The convergence theory of continued fraction and continued fraction methods in computational mathematics are two of the main areas of focus in current researches in continued fraction. The limit periodic continued fractions receive most attention in the convergence theory of continued fractions because most, if not all, instances of separate convergence occur for limit periodic continued fractions with elements that are functions of a complex variable.

In this section, we give only those basic definitions, notations and results pertaining to the theory of continued fractions that are used in the sequel.

Continued fractions and its various types

Definition 1.2.1. *Let* $\{a_n(z)\}_{n=1}^{\infty}$ *and* $\{b_n(z)\}_{n=1}^{\infty}$ *be two sequence of functions of a com-*

plex variable defined on a subset $\Omega \subseteq C$ *such that* $a_n(z) \not\equiv 0$ *for every* $n \geq 1$. *Define, for each* $n \geq 1$, *the transformation*

$$S_n(w) = \frac{a_n(z)}{b_n(z) + w}$$

and set,

$$f_n(z) = S_1 \circ S_2 \circ \cdots \circ S_n(0), \qquad n = 1, 2, \ldots$$

Then, the infinite triple $[\{a_n(z)\}_{n=1}^{\infty}\ \{b_n(z)\}_{n=1}^{\infty}\ \{f_n(z)\}_{n=1}^{\infty}]$ *is called a* **continued fraction.**

We denote a continued fraction by $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_n(z)}{b_n(z)}\right)$. Here, $a_n(z)$ and $b_n(z)$ are called the nth *partial numerators* and *partial denominators* of the continued fraction. The function $f_n(z)$ is called the nth *approximant* of the continued fraction.

Let $f_n(z) = \underset{i=1}{\overset{n}{\mathrm{K}}} \left(\frac{a_i(z)}{b_i(z)}\right) = \frac{A_n(z)}{B_n(z)}$ (*say*). The functions $A_n(z)$ and $B_n(z)$ are called the *numerator* and the *denominator* of nth approximant of the continued fraction. For each $n \geq 1$, the numerator $A_n(z)$ and the denominator $B_n(z)$ of nth approximant satisfy the following three term recurrence relation:

$$A_n(z) = b_n(z)\, A_{n-1}(z) + a_n(z)\, A_{n-2}(z) \qquad (1.2.1)$$

$$B_n(z) = b_n(z)\, B_{n-1}(z) + a_n(z)\, B_{n-2}(z) \qquad (1.2.2)$$

with initial functions

$$A_{-1} \equiv 1, \ \ A_0 \equiv 0, \ \ B_{-1} \equiv 0 \ \text{ and } \ B_0 \equiv 1\,.$$

The study of a number of special types of continued fractions is of importance for expanding analytic functions in terms of continued fractions. Some of the important types of continued fractions relevant to the present work are listed below.

- **C-fraction:** A continued fraction of the type

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_n\, z^{\alpha_n}}{1}\right), \qquad a_n \neq 0 \qquad (1.2.3)$$

is called a *C-fraction.* Here, α_n, $n = 1, 2, \ldots$, are positive integers and a_n, $n = 1, 2, \ldots$, are non-zero complex numbers.

- **Regular C-fraction:** A continued fraction of the type

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{a_n\, z}{1}\right), \qquad a_n \neq 0 \tag{1.2.4}$$

is called a *regular C-fraction.* Here, elements a_n, $n = 1, 2, \ldots$, are complex numbers.

- **General T-fraction:** A general T-fraction is a continued fraction of the form

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1 + G_n\, z}\right), \qquad F_n \neq 0\ . \tag{1.2.5}$$

where the elements F_n and G_n are complex numbers. If $F_n = 1$ for all n, then (1.2.5) is called a *T-fraction.*

We note that a regular C-fraction is a general T-fraction with $G_n = 0$ or a C-fraction with $\alpha_n = 1$, $n = 1, 2, \ldots$.

Remark 1.2.1. *We adopt the conventions that*

(i) a C-fraction, a general T-fraction and all of their approximants have the value zero at $z = 0$.

(ii) If an arbitrary function $b_0(z)$ *is added to a continued fraction of a particular type, then the new continued fraction remains of the same type.*

Several analytic functions are known to have representations in terms of regular C-fractions or general T-fractions. For example, various hypergeometric and confluent hypergeometric functions are represented by regular C-fractions or by general T-fractions [55]. Frequently a given function is represented by several different continued fractions, each with its own convergence behavior. Worpitzky [108], Sleszynski [90], Leighton and Scot [66] and others made significant contributions in the study of C-fractions. The T-fractions were

introduced by Thron [95] and extensively studied by Thron [54,95], Jones [54] and Waadeland [69]. The general T-fractions (1.2.5) were introduced by Perron [80] and its theory was further enriched by the works of Waadeland [102–105] and Jones and Thron [54] besides others.

Separate convergence of continued fractions

Let $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n(z)}{b_n(z)}\right)$ be a continued fraction, where the elements $a_n(z), b_n(z), n = 1, 2, 3, \ldots$ are analytic functions of a complex variable z in a domain D. Let $f_n(z)$ denote the nth approximant of the continued fraction. If the sequence $\{f_n(z)\}_{n=1}^{\infty}$ converges in a domain $\Omega \subseteq D$ to a finite value $f(z)$ for each $z \in \Omega$ then it is said that $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n(z)}{b_n(z)}\right)$ converges to $f(z)$ in Ω. A number of convergence results for various types of continued fractions can be found in [55].

In 1888, Sleszynski [90] showed that if the continued fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n z}{1}\right)$ satisfies $\sum |a_n| < \infty$ then not only does the sequence of approximants $\left\{\frac{A_n(z)}{B_n(z)}\right\}$ converges but also the sequences $\{A_n(z)\}$ and $\{B_n(z)\}$ converge separately to entire functions $A(z)$ and $B(z)$. The term *separate convergence* was introduced to describe this phenomena of continued fractions. In later investigations, the definition of separate convergence was modified for the general continued fractions as follows [77].

Definition 1.2.2. *Let* $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n(z)}{b_n(z)}\right)$ *be a continued fraction, where the elements in general are functions of a complex variable* z*. Let* $A_n(z)$ *and* $B_n(z)$ *denote the numerator and the denominator of nth approximant of the continued fraction. The continued fraction is said to* **converge separately** *for* $z \in \Omega$ *if there exists an "easily described" sequence* $\{\Gamma_n(z)\}$ *such that* $\left\{\frac{A_n(z)}{\Gamma_n(z)}\right\}$ *and* $\left\{\frac{B_n(z)}{\Gamma_n(z)}\right\}$ *both converge for* $z \in \Omega$*.*

Remark 1.2.2. *The restriction that the sequence* $\Gamma_n(z)$ *can be "easily described" is essential since, for convergent continued fractions, one can always choose* $\Gamma_n(z) = B_n(z)$ *and*

thus the distinction between ordinary and separate convergence would become meaningless.

The concept of separate convergence has recently attracted attention of several research workers mainly due to its applications especially in describing asymptotic behaviour of orthogonal polynomials [56, 57], in deriving results on analytic continuation of an analytic function and in describing behavior of the function on the boundary of the convergence region [52,53,67,82,83,99]. The elements of certain separately convergent continued fractions can provide the information about the growth of entire functions arising as its numerators (or denominators) [96].

Definition 1.2.3. *Let $a_n(z)$, $b_n(z)$, $a(z)$ and $b(z)$ be analytic functions of a complex variable z in a domain Ω. Further assume that $a_n(z) \neq 0$ and, for $z \in \Omega$,*

$$\lim_{n\to\infty} a_n(z) = a(z), \quad \lim_{n\to\infty} b_n(z) = b(z) .$$

Then,

$$\operatorname*{K}_{n=1}^{\infty} \left(\frac{a_n(z)}{b_n(z)}\right) \tag{1.2.6}$$

is said to be a **limit periodic continued fraction** *for $z \in \Omega$.*

Limit periodic continued fractions have been extensively studied, the reason being that so many useful continued fractions have this form and many times the separate convergence occur for limit periodic continued fractions with elements that are functions of a complex variable. The following result, due to Thron, is fundamental for finding the sufficient conditions for separate convergence of certain limit periodic continued fractions:

Theorem 1.2.1 ([97]). *Let*

$$\operatorname*{K}_{n=1}^{\infty} \left(\frac{a_n(z)}{b_n(z)}\right) \tag{1.2.7}$$

be a limit periodic continued fraction for $z \in \Omega$. Set

$$a_n(z) = a(z) + \delta_n(z), \qquad b_n(z) = b(z) + \eta_n(z)\,.$$

Let $x_1(z)$ and $x_2(z)$ be the solutions of

$$w^2 + b(z)\,w - a(z) = 0$$

and assume that the solutions have been so numbered that

$$\left|\frac{x_1(z)}{x_2(z)}\right| < 1 \qquad z \in \Omega^* \subset \Omega\,.$$

Further assume that the series

$$\sum_{k=1}^{\infty} |\delta_k(z)| \qquad \textit{and} \qquad \sum_{k=1}^{\infty} |\eta_k(z)|$$

converge uniformly on compact subsets of $\Omega_0 \subset \Omega$ and that

$$|x_2(z)| > 0 \qquad \textit{for } z \in \Omega_0\,.$$

Finally, let $\Omega^{(\epsilon)}$ be such that

$$|x_1(z) - x_2(z)| > 2\epsilon \qquad \textit{for } z \in \Omega^{(\epsilon)}\,.$$

Then,

$$\lim_{n\to\infty} \frac{A_n(z)}{(-x_2(z))^{n+1}} \qquad \textit{and} \qquad \lim_{n\to\infty} \frac{B_n(z)}{(-x_2(z))^{n+1}}$$

both exist and are analytic in

$$\Omega^\dagger = \bigcup_{\epsilon>0} \Omega^* \cap \Omega_0 \cap \Omega^{(\epsilon)}\,.$$

Here $A_n(z)$ and $B_n(z)$ are the numerator and denominator, respectively, of the nth approximant of the continued fraction (1.2.6).

Remark 1.2.3. *In Theorem* 1.2.1, *the continued fraction* $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n(z)}{b_n(z)}\right)$ *is limit periodic for* $z \in \Omega$. *This implies that* $\delta_n(z)$ *and* $\eta_n(z)$ *are analytic and*

$$\lim_{n\to\infty} \delta_n(z) = 0, \quad \lim_{n\to\infty} \eta_n(z) = 0 \qquad \textit{for } z \in \Omega\,.$$

The following result, due to Sleszynski for regular C-fractions, follows from the above theorem:

Corollary 1.2.1 ([90]). *If the regular C-fraction* $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n\,z}{1}\right)$, $a_n \neq 0$ *satisfies* $\sum_{n=1}^{\infty} |a_n| < \infty$ *then*

$$\lim_{n\to\infty} A_n(z) = A(z) \qquad \textit{and} \qquad \lim_{n\to\infty} B_n(z) = B(z)$$

for all $z \in \mathrm{C}$, *where* $A(z)$ *and* $B(z)$ *are entire functions. Here* $A_n(z)$ *and* $B_n(z)$ *are the numerator and denominator, respectively, of the nth approximant of the regular C-fraction.*

Remark 1.2.4. *It is observed that Corollary* 1.2.1 *is the special case of Theorem* 1.2.1 *with* $a(z) \equiv 0$, $b(z) \equiv 1$, $\delta_n(z) = a_n z$, $\eta_n(z) \equiv 0$, $x_1(z) \equiv 0$, $x_2(z) \equiv -1$. *For, with this particular choices,* $\Omega = \Omega^* = \Omega_0 = \mathrm{C}$ *and* $\Omega^{(\epsilon)} = \mathrm{C}$, $0 < \epsilon < 1$.

The following result, due to Schwartz also follows from Theorem 1.2.1:

Corollary 1.2.2 ([84]). *For the continued fraction*

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{k_n}{z - c_n}\right)$$

let, $\sum_{n=1}^{\infty} |k_n| < \infty$, $\sum_{n=1}^{\infty} |c_n - c| < \infty$ *hold. Then,*

$$\lim_{n\to\infty} \frac{A_n(z)}{(c-z)^{n+1}} = C^*(z), \qquad \lim_{n\to\infty} \frac{B_n(z)}{(c-z)^{n+1}} = D^*(z)$$

exist and the functions $C^*(z)$ *and* $D^*(z)$ *are analytic in* $\mathrm{C} \setminus \{c\}$.

Remark 1.2.5. *In Corollary* 1.2.2, *the special type of continued fraction* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{k_n}{z-c_n}\right)$ *is known as a J-fraction. By setting* $x_1(z) \equiv 0$, $x_2(z) \equiv z-c$, $\Omega = \Omega^* = \mathbb{C}$, $\Omega_0 = \mathbb{C} \setminus \{c\}$, $\Omega^{(\epsilon)} = \{z \in \mathbb{C} : |z-c| > 2\epsilon\}$, $0 < \epsilon < 1$, *the Corollary* 1.2.2 *easily follows from Theorem* 1.2.1.

The following result for general T-fractions also follows from Theorem 1.2.1:

Corollary 1.2.3. *Let* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$, *be a general T-fraction. Let* $\sum_{n=1}^{\infty} |F_n| < \infty$ *and* $\sum_{n=1}^{\infty} |G_n - G| < \infty$, $G \neq 0$. *Then,*

$$\lim_{n\to\infty} \frac{A_n(z)}{(1+Gz)^{n+1}} = C(z) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{B_n(z)}{(1+Gz)^{n+1}} = D(z)$$

exist for all $z \in \mathbb{C} \setminus \{\frac{-1}{G}\}$. *Further, the functions* $C(z)$ *and* $D(z)$ *are analytic in* $\mathbb{C} \setminus \{\frac{-1}{G}\}$.

Remark 1.2.6. *By setting* $x_1(z) \equiv 0$, $x_2(z) \equiv -1-Gz$, $\Omega = \mathbb{C}$, $\Omega^* = \mathbb{C} \setminus \{\frac{-1}{G}\} = \Omega_0$, $\Omega^{(\epsilon)} = \{z \in \mathbb{C} : |1+Gz| > 2\epsilon\}$, $0 < \epsilon < 1$, *Corollary* 1.2.3 *easily follows from Theorem* 1.2.1.

Corollary 1.2.4. *Let* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$, *be a general T-fraction. Let*

$$\lim_{n\to\infty} F_n = 1 \qquad \lim_{n\to\infty} G_n = -1$$

and $\sum_{n=1}^{\infty} |F_n - 1| < \infty$ *and* $\sum_{n=1}^{\infty} |G_n + 1| < \infty$. *Then,*

$$\lim_{n\to\infty} A_n(z) = A(z) \qquad \text{and} \qquad \lim_{n\to\infty} B_n(z) = B(z)$$

for $|z| < 1$ *and the functions* $A(z)$ *and* $B(z)$ *are analytic exist for* $|z| < 1$. *Also*

$$\lim_{n\to\infty} \frac{A_n(z)}{z^{n+1}} = C(z) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{B_n(z)}{z^{n+1}} = D(z)$$

exist for $|z| > 1$ *and the functions* $C(z)$ *and* $D(z)$ *are analytic for* $|z| > 1$.

Remark 1.2.7. *Corollary* 1.2.4 *for* $|z| < 1$, *follows from Theorem* 1.2.1 *by setting* $x_1(z) \equiv -z$, $x_2(z) \equiv -1$, $\Omega = \mathbb{C}$ $\Omega^* = \{z \in \mathbb{C} : |z| < 1\}$ $\Omega^{(\epsilon)} = \{z \in \mathbb{C} : |z-1| > 2\epsilon\}$, $0 < \epsilon < 1$ *while, for* $|z| > 1$, *it follows by setting* $x_1(z) \equiv -1$, $x_2(z) \equiv -z$, $\Omega = \mathbb{C}$ $\Omega^* = \{z \in \mathbb{C} : |z| > 1\}$, $\Omega^{(\epsilon)} = \{z \in \mathbb{C} : |1-z| > 2\epsilon\}$.

Thron proved the following:

Theorem 1.2.2 ([98]). *Let* $\overset{\infty}{\underset{n=1}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$, *be a general T-fraction satisfying* $\sum_{n=1}^{\infty} |F_n| < \infty$ *and* $\sum_{n=1}^{\infty} |G_n| < \infty$. *Then,*

$$\lim_{n\to\infty} A_n(z) = A(z) \qquad \text{and} \qquad \lim_{n\to\infty} B_n(z) = B(z)$$

exist for all $z \in \mathbb{C}$ *and* $A(z)$, $B(z)$ *are entire functions, where* $A_n(z)$ *and* $B_n(z)$, $n = 1, 2, \ldots$, *are the numerators and the denominators, respectively, of the approximants of the general T-fraction.*

1.3 Growth aspects of entire functions

The growth of entire functions, as measured by their maximum modulii, plays a very important role in the study of their asymptotic values, exceptional values, distribution of zeros, complex dynamics etc. In this section, we review some basic results concerning the growth of entire functions that are used in the sequel.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of the complex variable $z = re^{i\theta}$ and let

$$M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)| \, .$$

The function $M(r,f)$ is called the maximum modulus of $f(z)$ for $|z| = r$.

Definition 1.3.1. *An entire function* $f(z)$ *is said to be of finite order if there exists a constant* A *such that*

$$M(r,f) < \exp\left(r^A\right)$$

for all sufficiently large values of r. The greatest lower bound $\rho(f)$ of all such numbers A is called the **order** *of the function $f(z)$. Thus,*

$$\rho \equiv \rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} .$$

If no constant A can be found such that $M(r,f) < \exp\left(r^A\right)$ holds, then $f(z)$ is said to be of infinite order and such functions are said to be of *fast growth.* The entire functions of zero order are said to be of *slow growth.*

To compare the rate of growth of entire functions with the same non-zero finite order, the concept of type has been introduced as follows:

Definition 1.3.2. *An entire function $f(z)$, having non-zero finite order $\rho(f)$, is said to be* **type** *$T(f)$ if*

$$T \equiv T(f) = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho(f)}} .$$

According as $T = \infty$, $0 < T < \infty$ or $T = 0$, $f(z)$ is said to be of *maximal, mean* or *minimal type* of order ρ. An entire functions $f(z)$ is said to have *growth* (ρ, T) if its order does not exceed ρ and its type does not exceed T if it is of order ρ. An entire function of growth $(1,T)$, $T < \infty$ is called a function of *exponential type.*

For an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ to be of finite order and finite type, necessary and sufficient conditions, in terms of its Taylor coefficients a_n, have been found [23]. Thus, the entire function $f(z)$ is of finite order $\rho(f)$ if and only if

$$\rho(f) = \limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|} < \infty \tag{1.3.1}$$

Further, the entire function $f(z)$ is of order $\rho(f)$, $0 < \rho(f) < \infty$ and type $T(f)$, $0 < T(f) < \infty$, if and only if,

$$e\,\rho(f)\,T(f) = \limsup_{n\to\infty} n|a_n|^{\frac{\rho(f)}{n}} \tag{1.3.2}$$

For a more precise specification of the rate of growth of $f(z)$, Whittaker [107] introduced the concept of lower order for an entire function. Thus,

Definition 1.3.3. *An entire function* $f(z)$ *is said to be of* **lower order** $\lambda(f)$ $(\lambda(f) \leq \rho(f))$ *if*

$$\lambda \equiv \lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}\,.$$

The characterizations for the lower order $\lambda(f)$ of an entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in terms of the Taylor coefficients a_n, $n = 0, 1, \ldots$ have been found by Shah [86, 87], Juneja [60] and Juneja and Kapoor [61].

If the entire function $f(z)$ is of infinite or zero order the definition of type is not feasible and so the growth of such functions cannot be compared precisely by confining to the concept of order only. For studying the growth of such functions, the notion of generalized orders is introduced. The following classes L^0, Λ, Ω and $\overline{\Omega}$ are defined for this purpose:

Let L^0 denote the class of functions $h(x)$ satisfying (A,i) and (A,ii):

(A,i) $h(x)$, defined on $[a,\infty)$, is positive, strictly increasing, differentiable, and tends to ∞ as $x \to \infty$.

(A,ii)
$$\lim_{x\to\infty} \frac{h[x(1+\tilde{g}(x))]}{h(x)} = 1\,,$$

for every function $\tilde{g}(x)$ such that $\tilde{g}(x) \to 0$ as $x \to \infty$.

Let Λ denote the class of functions $h(x)$ satisfying (A,i) and (A,iii):

(A,iii)
$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1 \qquad \text{for every } c, \qquad 0 < c < \infty.$$

Let Ω denote the class of functions $h(x)$ satisfying (A,i) and (A,iv):

There exists a $\delta(x) \in \Lambda$ and x_0, K_1 and K_2 such that

(A,iv) $$0 < K_1 \leq \frac{d(h(x))}{d(\delta(\log x))} \leq K_2 < \infty \qquad \text{for all } x > x_0.$$

Let $\overline{\Omega}$ denote the class of functions $h(x)$ satisfying (A,i) and (A,v):

(A,v) $$\lim_{x \to \infty} \frac{d(h(x))}{d(\log x)} = K, \qquad 0 < K < \infty.$$

In the sequel, wherever necessary, we shall assume that $h(x) \in L^0$ can be extended over $(-\infty, \infty)$ by the definition $h(x) = h(a)$ *for* $x \in (-\infty, a)$.

Let

$$f(z) = \sum_{n=0}^{\infty} c_n \, z^{\lambda_n} \tag{1.3.3}$$

be a nonconstant entire function. Here $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive integers such that no element of the sequence $\{c_n\}_{n=1}^{\infty}$ is zero.

Serementa [85] and Shah [88] introduced the concepts of generalized (α, β)- order and generalized lower (α, β)- order of an entire function $f(z)$ as follows using more general functions than the logarithmic function to compare the growth of $\log M(r, f)$ with that of $\log r$:

Definition 1.3.4. *The generalized (α, β)- order $\rho(\alpha, \beta; f)$ of an entire function $f(z)$, given by (1.3.3), is defined as*

$$\rho(\alpha, \beta; f) = \limsup_{r \to \infty} \frac{\alpha(\log M(r, f))}{\beta(\log r)} \tag{1.3.4}$$

where $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$ and $M(r, f) = max_{|z|=r} |f(z)|$.

Definition 1.3.5. *The generalized lower (α, β)- order $\lambda(\alpha, \beta; f)$ of an entire function $f(z)$, given by (1.3.3), is defined as*

$$\lambda(\alpha, \beta; f) = \liminf_{r \to \infty} \frac{\alpha(\log M(r, f))}{\beta(\log r)} \tag{1.3.5}$$

where $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$ *and* $M(r, f) = max_{|z|=r}|f(z)|$.

Let $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x} = O(1)$*as* $x \to \infty$ for all c, $0 < c < \infty$, and $f(z)$, defined by (1.3.3), be an entire function having generalized order $\rho(\alpha\,,\,\beta\,;f)$. Then, a characterization of $\rho(\alpha\,,\,\beta\,;f)$ in terms of the coefficients of Taylors series of $f(z)$ is given as [85]

$$\rho(\alpha\,,\,\beta\,;f) = \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta(\frac{1}{\lambda_n}\log|c_n|^{-1})}\,. \tag{1.3.6}$$

Further [88],

$$\lambda(\alpha, \beta; f) \geq \liminf_{n\to\infty} \frac{\alpha(n)}{\beta\left(\frac{1}{n}\log|c_n|^{-1}\right)} \tag{1.3.7}$$

and, if $\Psi(n) = \left|\frac{a_n}{a_{n+1}}\right|$ is ultimately a non-decreasing function of n, the function $\alpha(x)$, $\beta(x)$ satisfy

$$\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x} = O(1) \qquad \text{as} \quad x \to \infty$$

and there exists a function $\eta(x)$ satisfying $\eta(x) \to \infty$ as $x \to \infty$ such that

$$\frac{\beta(x\eta(x))}{\beta(e^x)} \to 0 \qquad \text{as} \quad x \to \infty$$

then

$$\lambda(\alpha, \beta; f) = \liminf_{n\to\infty} \frac{\alpha(n)}{\beta\left(\frac{1}{n}\log|c_n|^{-1}\right)}\,. \tag{1.3.8}$$

The concept of generalized $(\alpha, \beta; f)$- order and generalized lower $(\alpha, \beta; f)$- order are inadequate to give any specific information about the growth of the class of entire functions of slow growth. To measure the growth of such entire functions satisfactorily, the notions of generalized (α, α)- order and lower (α, α)- order are introduced by Kapoor and Nautiyal [62] as follows:

Definition 1.3.6 ([62]). *The generalized* $(\alpha\,,\,\alpha)$*- order* $\rho(\alpha\,,\,\alpha\,;f)$ *of an entire function* $f(z)$ *, given by (*1.3.3*), is defined as*

$$\rho(\alpha\,,\,\alpha\,;f) = \limsup_{r\to\infty} \frac{\alpha(\log M(r,f))}{\alpha(\log r)} \tag{1.3.9}$$

where $\alpha(x)$ *either belongs to* Ω *or* $\overline{\Omega}$ *and* $M(r, f) = max_{|z|=r}|f(z)|$.

Since $\log M(r, f)$ is a convex function of $\log r$, it follows that $\rho(\alpha, \alpha; f) \geq 1$. The characterization of $\rho(\alpha, \alpha; f)$ in terms of the coefficients of Taylors series of $f(z)$ is known ([62], Theorem 4). Thus, let $f(z)$, defined by (1.3.3), be an entire function having generalized order $\rho(\alpha, \alpha; f) \equiv \rho\ (1 \leq \rho \leq \infty)$. Then,

$$\rho(\alpha, \alpha; f) = P(\widetilde{L}) = \begin{cases} max\{1, \widetilde{L}\} & if\ \alpha(x) \in \Omega \\ 1 + \widetilde{L} & if\ \alpha(x) \in \overline{\Omega} \end{cases} \tag{1.3.10}$$

where,

$$\widetilde{L} = \limsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\alpha(\frac{1}{\lambda_n}\log|c_n|^{-1})}\ . \tag{1.3.11}$$

Definition 1.3.7. *The generalized lower* (α, α)*-order* $\lambda(\alpha, \alpha; f)$ *of an entire function* $f(z)$*, given by (*1.3.3*), is defined as*

$$\lambda(\alpha, \alpha; f) = \liminf_{r\to\infty} \frac{\alpha(\log M(r, f))}{\alpha(\log r)} \tag{1.3.12}$$

where $\alpha(x)$ *either belongs to* Ω *or* $\overline{\Omega}$ *and* $M(r, f) = max_{|z|=r}|f(z)|$.

The characterisation of $\lambda(\alpha, \alpha; f)$ in terms of the coefficients of Taylors series of $f(z)$ is also known [62]. Let $f(z)$, defined by (1.3.3), be an entire function having generalized lower order $\lambda(\alpha, \alpha; f) \equiv \lambda(1 \leq \lambda \leq \infty)$ and $\{n_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers. Then,

$$\lambda(\alpha, \alpha; f) = P_\chi(\widetilde{l}) = \begin{cases} max\{1, \widetilde{l}\} & if\ \alpha(x) \in \Omega \\ \chi + \widetilde{l} & if\ \alpha(x) \in \overline{\Omega} \end{cases} \tag{1.3.13}$$

where,

$$\widetilde{l} \equiv l(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\frac{1}{\lambda_{n_k}}\log|c_{n_k}|^{-1})} \tag{1.3.14}$$

and $\chi \equiv \chi(\{n_k\}) = \liminf_{k\to\infty} \frac{\alpha(\lambda_{n_{k-1}})}{\alpha(\lambda_{n_k})}$.

Let $f(x)$ be a restriction to $[-1,-1]$ of an entire function $f(z)=\sum_{n=0}^{\infty} a_n z^n$ and let for $n=0,1,\ldots$

$$E_n(f)=\inf_{p\in\pi_n}||f-p|| \tag{1.3.15}$$

where, $||f-p|| = \sup_{-1\le x\le 1}|f(x)-p(x)|$ and π_n denotes the set of all polynomials p of degree atmost n.

The rate of decay of the approximation error $E_n(f)$ as influenced by the generalized (α,β)- order $\rho(\alpha,\beta;f)$ and the generalized lower (α,β)- order $\lambda(\alpha,\beta;f)$ is given by the following:

Theorem 1.3.1 ([88]). *Let $f(x)$ be the restriction to $[-1,1]$ of an entire function $f(z)$ and let $E_n(f)$ be defined by (1.3.15). Let $\alpha(x)\in L^0$ and $\beta(x)\in\Lambda$. Then,*

$$\lambda(\alpha,\beta;f)\ge \liminf_{n\to\infty}\frac{\alpha(n)}{\beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)}. \tag{1.3.16}$$

Further, if $\frac{dF(x,c)}{d(\log x)}=O(1)$, $x\to\infty$, for every $c>0$ and $\beta(x\psi(x))/\beta(e^x)\to 0$, as $x\to\infty$ for some function $\psi(x)$ tending to ∞ as $x\to\infty$, then

$$\rho(\alpha,\beta;f)=\limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)}. \tag{1.3.17}$$

Assume further that $E_n(f)/E_{n+1}(f)$ is ultimately a nondecreasing function of n. Then, only the equality sign holds in (1.3.16).

An analogue of Theorem 1.3.1 for generalized (α,α)- order $\rho(\alpha,\alpha;f)$ and the generalized lower (α,α)- order $\lambda(\alpha,\alpha;f)$ is due to Kapoor and Nautiyal. Thus,

Theorem 1.3.2 ([62]). *Let $f(x)$ be the restriction to $[-1,1]$ of an entire function $f(z)$ and let $E_n(f)$ be defined by (1.3.15). Let $\alpha(x)\in\Omega$ or $\overline{\Omega}$ and $P(\Psi)=\max\{1,\Psi\}$ if $\alpha(x)\in\Omega$, $P(\Psi)=1+\Psi$ if $\alpha(x)\in\overline{\Omega}$. Then,*

(i) $\rho(\alpha,\alpha;f)=P(L)$, where

$$L=\limsup_{n\to\infty}\frac{\alpha(n)}{\alpha\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)}. \tag{1.3.18}$$

(ii) $\lambda(\alpha, \alpha; f) \geq P(l)$, *where*

$$l = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)} . \tag{1.3.19}$$

(iii) Further, if $E_n(f)/E_{n+1}(f)$ *is ultimately decreasing, then* $\lambda(\alpha, \alpha; f) = P(l)$.

A different approach [47] to establish the relations between the maximum modulus and the Taylors coefficients is found in the form of direct estimates instead of measuring growth of the entire function $f(z)$ by its order, type or generalized orders. This approach is used as well to find the characterization of the rate of decay of the error in the polynomial approximation of an entire function in terms of the rate of growth of its sequence of Taylor coefficients [46].

Growth aspects of numerator and denominator of continued fractions

Let $\underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{a_n z}{1}\right)$, $a_n \neq 0$, $n \geq 1$ be a regular C-fraction (c.f. Section 1.2). Let $A_n(z)$ and $B_n(z)$ denote respectively the numerator and denominator of the nth approximant of the regular C-fraction. Sleszynski [90] showed that if the regular C-fraction satisfies the condition $\sum_{n=1}^{\infty} |a_n| < \infty$ then the sequence $\{A_n(z)\}_{n=1}^{\infty}$ and $\{B_n(z)\}_{n=1}^{\infty}$ converge uniformly on compact subsets of C to entire functions $A(z)$ and $B(z)$ respectively.

Since $B_n(z)$ is a polynomial of degree $[\frac{n}{2}]$, set $B_n(z) = \sum_{k=0}^{[\frac{n}{2}]} q_k^{(n)} z^k$. It is known that $B_n(z)$ satisfies the recursion relation $B_n(z) = B_{n-1}(z) + a_n zB_{n-2}(z)$, $B_0(z) = 1$, $B_1(z) = 1$. Further, it follows easily that

$$q_k^{(n)} = \sum_{m_1=2}^{n-2(k-1)} \sum_{m_2=m_1+2}^{n-2(k-2)} \cdots \sum_{m_k=m_{k-1}+2}^{n} a_{m_1} a_{m_2} \cdots a_{m_{k-1}} a_{m_k}, \qquad n > 2k \tag{1.3.20}$$

and that the following Śleszyński's inequalities [90] hold:

$$|q_k^{(n)}| \leq \frac{\left(\sigma_2^{(n)}\right)^k}{k\,!}, \qquad n > 2k \tag{1.3.21}$$

$$|q_k^{(n)}| \leq \prod_{m=1}^{k} \sigma_{2m}^{(n)}, \qquad n > 2k \tag{1.3.22}$$

where, $\sigma_m^{(n)} = \sum_{k=m}^{n} |a_k|$ and $\sigma_m = \sum_{k=m}^{\infty} |a_k|$. Also, it is easy to establish the equiconvergence (not depending on k) of the family of sequences $\left\{q_k^{(n)}\right\}_{n=1}^{\infty}$ to q_k and uniform convergence of $B_n(z)$ to $B(z)$ on compact subsets of C, so that

$$\lim_{n\to\infty} B_n(z) = B(z) = \sum_{k=0}^{\infty} q_k\, z^k \tag{1.3.23}$$

Using (1.3.21), it follows that $|B(z)| \le e^{\sigma_2 |z|}$. Thus, for a regular C-fraction satisfying $\sum |a_n| < \infty$, $B(z)$ is an entire function of exponential type.

The following result describes the effect of the sequence $\{a_n\}_{n=1}^{\infty}$ on the order of $A(z)$ and $B(z)$.

Theorem 1.3.3 ([70]). *Let* $\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n\, z}{1}\right)$, $a_n \ne 0$, $n \ge 1$ *with* $\sum_{n=1}^{\infty} |a_n| < \infty$ *be a regular C-fraction. If* $a_n \ge c/n^{\mu}$, $c > 0$, $\mu > 1$, $n \ge 1$, *then the order of* $A(z)$ *and* $B(z)$ *are not less than* $(1/\mu)$, *where* $A(z)$ *and* $B(z)$ *are the limit functions of the sequences of the numerators and denominators of approximants of the regular C-fraction respectively.*

Thron further obtained the following complementary results on the order and type of $A(z)$ and $B(z)$.

Theorem 1.3.4 ([96]). *Let* $\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{a_n\, z}{1}\right)$, $a_n \ne 0$, $n \ge 1$ *with* $\sum_{n=1}^{\infty} |a_n| < \infty$ *be a regular C-fraction. Let* $A_n(z)$ *and* $B_n(z)$ *denote respectively the numerator and denominator of the nth approximant of the regular C-fraction. Then, the sequence* $\{A_n(z)\}_{n=1}^{\infty}$ *and* $\{B_n(z)\}_{n=1}^{\infty}$ *converge uniformly on compact subsets of* C *to entire functions* $A(z)$ *and* $B(z)$ *of order atmost one respectively. Further,*

(1) if $|a_n| \le r^n$, $0 < r < 1$, $n \ge 1$, *then* $A(z)$ *and* $B(z)$ *are of order zero.*

(2) if for $c > 0$ *and* $\mu > 1$, $|a_n| \le c/n^{\mu}$, $n \ge 1$, *then* $A(z)$ *and* $B(z)$ *are of order at most* $(1/\mu)$. *if* $A(z)$ *(or* $B(z)$*) is of order* $(1/\mu)$, *then it is of type at most* $\frac{\mu}{e^{\tau}}\left(\frac{2\,c}{(\mu-1)\tau^3}\right)^{\frac{1}{\mu}}$, *where* τ *is the solution of the equation* $2(1-\tau)^{\mu+2} = (\mu-1)\tau^3$.

(3) if $a_n = 1/n(\log n)^{\alpha}$, $\alpha > 1$, *then the functions* $A(z)$ *and* $B(z)$ *are of order one. If the order of* $A(z)$ *(or* $B(z)$*) is one, then it is of minimum type.*

Combining Maillet's result with the above result, Thron concluded that if $a_n = 1/n^{\mu}$, then the functions $A(z)$ and $B(z)$ are of order $(1/\mu)$.

For a general T-fraction (c.f. Section 1.2) also, the growth of $A(z)$ and $B(z)$ is studied by Thron. Thus,

Theorem 1.3.5 ([98]). *Let* $\overset{\infty}{\underset{n=1}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$, *be a general T-fraction satisfying* $\sum_{n=1}^{\infty}|F_n| < \infty$ *and* $\sum_{n=1}^{\infty}|G_n| < \infty$. *Let* $A_n(z)$ *and* $B_n(z)$ *denote the numerator and denominator of the nth approximant of the general T-fraction respectively. Then, the sequences* $\{A_n(z)\}$ *and* $\{B_n(z)\}$ *converge uniformly on each compact subset of* $\mathbb{C}$ *to the entire functions* $A(z)$ *and* $B(z)$ *having order atmost one.*

Growth aspects in complex analytic dynamics

In the dynamics of an entire transcendental function, the order and the type of an entire function play an important role. For a polynomial, an entire function of order zero, the basin of attraction of any finite attracting periodic point is bounded. However, this is not necessarily true for entire transcendental functions. Bhattacharyya [21] showed that the basin of attraction of any finite attracting periodic point is bounded if the entire transcendental function has growth $(\frac{1}{2}, 0)$ (c.f. Section 1.3). Thus,

Theorem 1.3.6 ([21]). *Let* $f(z)$ *be a nonconstant entire function of growth* $(\frac{1}{2}, 0)$ *and let* $\alpha \in \mathbb{C}$ *be an attracting periodic point of period n. Then, the basin of attraction of* α *is bounded.*

The estimates for the growth of functions with unbounded (e.g. including half planes) basins of attraction are obtained in [21].

Baker proved that if $f(z)$ is an entire transcendental function of sufficiently small rate of growth, then $\mathcal{F}(f)$ can have no unbounded completely invariant component (c.f. Definition 1.1.6) and under suitable slow growth conditions no unbounded component at all. Thus,

Theorem 1.3.7 ([9]). *If for a transcendental entire $f(z)$ there is an unbounded invariant component of the Fatou set, then $f(z)$ must be of growth greater than $(\frac{1}{2}, 0)$.*

Theorem 1.3.8 ([9]). *If a transcendental entire function $f(z)$ is of generalized (α, α)-order p with $\alpha(x) \equiv \log x$, and $1 < p < 3$, then every component of the Fatou set of $f(z)$ is bounded.*

Theorem 1.3.9 ([9]). *If $f(z)$ is a transcendental entire function of growth not greater than $(\frac{1}{2}, 0)$, then the Fatou set $\mathcal{F}(f)$ has no completely invariant component.*

There is a close correlation between the order of an entire function and the number of asymptotic values of the function. It is well known that the asymptotic values of an entire function play a vital role in the dynamics. In 1907, Denjoy [24] conjectured that an entire function of order ρ can have atmost 2ρ asymptotic values. The conjecture was proved by in affirmative by Ahlfors, in 1932. Thus, we have the following Denjoy-Ahlfors Theorem.

Theorem 1.3.10 ([1]). *If an entire function is of order ρ then it has atmost 2ρ different asymptotic values.*

1.4 Present work

Motivation

The chaotic dynamics and fractals have become quite popular in recent years due to its wide ranging application in engineering problems. The *complex analytic dynamics* is an intricate and fascinating area of dynamical systems in which *deterministic fractals* appear often as a chaotic sets. During the last decade there has been a renewed interest in the dynamics of analytic functions due to the beautiful computer graphics related to it.

The central objects studied in complex analytic dynamics of a function are its Julia set and Fatou set. There are two basic approaches in the study of dynamics of a function.

The first one is to investigate the iterative behaviour of an individual function, while the second one is the study of the iterative behaviour changes due to slight perturbations in the function. In the latter approach, which has received considerable attention during recent years, the simplest (but sufficiently intricate) case being that of a family of functions that depends on one parameter.

The dynamics of an entire transcendental function is much more interesting than the dynamics of a polynomial, since in the case of an entire transcendental function substantial hyperbolicity occurs in dynamics. The Julia set of a polynomial is always bounded. But, it is obvious from Picard's theorem that the Julia set of an entire transcendental function is unbounded, so that its Fatou set no longer forms a neighborhood of ∞. In the dynamics of polynomials, the basin of attraction of any finite attracting periodic point is bounded. But, in the dynamics of entire transcendental functions, the basins of attraction of finite attracting periodic points may become unbounded. The Julia sets of certain entire transcendental functions are often Cantor bouquets giving beautiful examples of *fractals*. Devaney and Durkin [33] observed the burst nature in the Julia set of the exponential function. If $0 < \lambda < (1/e)$, the chaotic region for the function $\lambda \exp(z)$ is a nowhere dense set entirely contained in the right half plane, while if $\lambda > (1/e)$ the chaotic region is the entire complex plane. This phenomena is referred to as *explosion* (or *chaotic burst*) in the Julia sets of functions in one parameter family of functions $\mathcal{E} = \{\lambda \exp(z) : \lambda > 0\}$. Devaney [25, 28] observed similar explosion in the Julia sets of functions in the family $\mathcal{C} = \{i\,\lambda\, \cos z \,:\, \lambda > 0\}$. This kind of explosion in the chaotic set does not occur in the dynamics of a polynomial. Further, certain new types of stable domains like a wandering domain [8] and a domain at infinity [42] exist for transcendental entire functions but are not found for polynomials.

In the dynamics of entire functions, the dynamics of polynomials and dynamics of certain classes of transcendental entire functions are hitherto studied by taking advantage

of the presence of finitely many critical values and asymptotic values of their functions. The dynamical behaviour of critically finite (*i.e.*, having only finitely many critical values and asymptotic values) entire transcendental functions share many of the properties of polynomials and rational functions; for instance, these functions do not have wandering domains. Exploiting the critical finiteness, Devaney and coworkers studied exhaustively the dynamics of some of the most interesting periodic entire transcendental functions like λe^z, $\lambda \sin z$ and $\lambda \cos z$. However, the dynamics of non-critically finite entire functions has not been explored so far, probably because of non-applicability of Sullivan's theorem (c.f. Theorem 1.1.6) to these functions. Also, the presence of infinitely many critical values and the behaviour of the orbits of critical values make it difficult to study the dynamics of non-critically finite entire functions. In the present work an effort is made in this direction.

The non-critically finite entire functions considered here for the study are obtained as numerators or denominators of certain separately convergent continued fractions and include entire functions like $(e^z-1)/z$, $\sinh z/z$, the modified Bessel function $I_0(z)$ of order zero.

Organization

The present work is organized into six chapters; Chapter 1 being the Introduction. gives a brief review of the basic theory and results relevant to our study in the subsequent chapters.

Chapter 2

In Chapter 2, the growth of the entire functions arising as the numerator and the denominator of a separately convergent continued fraction is studied. Let $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$, $n \geq 1$ be a general T-fraction satisfying the conditions $\sum_{n=1}^{\infty} |F_n| < \infty$ and $\sum_{n=1}^{\infty} |G_n| < \infty$. Let $A_n(z)$ and $B_n(z)$ denote respectively the numerator and the denominator of the nth approximant of the general T-fraction and $\lim_{n\to\infty} A_n(z) = A(z)$, $\lim_{n\to\infty} B_n(z) = B(z)$ uniformly on compact subsets of $\mathbb{C}$. In the present chapter, the

growth of the entire functions $A(z)$ and $B(z)$ is studied by investigating the influence of the elements F_n and G_n of a general T-fraction on the order of $A(z)$ and $B(z)$. In Section 2.1, It is proved that if F_n and G_n tend to zero sufficiently rapidly then the entire functions $A(z)$ and $B(z)$ are of order zero. This result generalizes a result of Thron [98] obtained for a regular C-fraction $\underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{F_n\, z}{1}\right)$. An attempt to find the influence of the elements of a general T-fraction on the generalized (α, α)- order of its numerator and denominator is made in Section 2.2. It is seen that our results do give significant information about the comparison of growth of $B(z)$ (or $A(z)$) even for regular C-fractions in the situations where the relevant result of Thron [98] does not give any nontrivial information. The results pertaining to generalized (α, α)- order of $A(z)$ and $B(z)$ are used in Chapter 3 to study the dynamics of certain entire transcendental functions of slow growth. The results concerning influence of the elements of a general T-fraction on the generalized (α, β)- order of $A(z)$ and $B(z)$ are found in Section 2.3 and one of our results in this section, giving a sufficient condition on the elements F_n, G_n that forces the numerator (*or* denominator) of a general T-fraction to have generalized $(\alpha\,, \beta)$- order not less than a prespecified constant, generalizes a result of Maillet [70]. Finally, since even entire functions can not be obtained as a denominator (*or* numerator) of a separately convergent general T-fraction, a new type of continued fraction, called modified general T-fraction, is introduced in Section 2.4 and the growth of the numerator and the denominator of such a modified general T-fraction when they are slow growth, is studied.

Chapter 3

Chapter 3 is devoted to the study of the dynamics of slow growth entire functions arising as the numerator $A(z)$ and the denominator $B(z)$ of separately convergent general T-fractions having generalized (α, α)- order μ with $\alpha(x) \equiv \log x$ and $2 < \mu < 3$. For this purpose, one parameter families $\mathcal{A} \equiv \{A_\lambda(z) = \lambda A(z) : \lambda > 0\}$ and $\mathcal{B} \equiv \{B_\lambda(z) = \lambda B(z) : \lambda > 0\}$ are

considered. In Section 3.2, the dynamics of $A_\lambda \in \mathcal{A}$, $\lambda > 0$, is studied. In particular, the nature of the fixed points of $A_\lambda(z)$ on the positive real line is investigated and the dynamics of $A_\lambda(x)$ for $x \geq 0$ is described. Further, in this section, the dynamics of $A_\lambda(z)$ for $z \in \mathbb{C}$ is described for the three different cases, viz, $0 < \lambda < \lambda_A^*$, $\lambda = \lambda_A^*$ and $\lambda > \lambda_A^*$ where $\lambda_A^* = \frac{1}{A'(0)}$. In all the three cases, we obtain computationally useful characterization of the Julia set of $A_\lambda(z)$ as the closure of the set of points with orbits escaping to infinity under iteration of A_λ. Such a characterization was hitherto known only for critically finite entire transcendental functions [37]. In Section 3.3, firstly, the nature of the fixed points of $B_\lambda(z)$ and the dynamics of $B_\lambda(z)$ on the positive real line are investigated. Next, a description of the basin of attraction (c.f. Theorem 1.1.7) of the real attracting fixed point a_λ of the entire function $B_\lambda(z)$ is found for $0 < \lambda < \lambda_B^* = \frac{1}{B'(x^*)}$; x^* being the unique positive real root of the equation $B(x) - xB'(x) = 0$. Similarly, a description of the parabolic domain (c.f. Theorem 1.1.7) corresponding to the rationally indifferent fixed point x^* of $B_\lambda(z)$ is found for $\lambda = \lambda_B^*$. Finally, in this section, the dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ is described for all the three different cases, viz, $0 < \lambda < \lambda_B^*$, $\lambda = \lambda_B^*$ and $\lambda > \lambda_B^*$ and, analogous to that of $A_\lambda(z)$, a computationally useful characterization of the Julia set of $B_\lambda(z)$ is obtained. Finally, in Section 3.4, the characterizations of the Julia sets of $A_\lambda(z)$ and $B_\lambda(z)$, obtained in Sections 3.2 and 3.3, are applied to computationally generate the pictures of the Julia sets of $A_\lambda \in \mathcal{A}$ and $B_\lambda \in \mathcal{B}$ for different values of λ.

Chapter 4

Let $f(z) = (e^z - 1)/z$ be the non-critically finite entire function arising as the denominator of the separately convergent general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{z/(n+1)}{1-(z/(n+1))} \right)$. In Chapter 4, the dynamics of the entire function $f_\lambda(z) = \lambda f(z)$, $\lambda > 0$ is studied. Let $\mathcal{K} \equiv \{f_\lambda(z) = \lambda f(z) : \lambda > 0\}$ be one parameter family of functions. In Section 4.2, some of the basic properties of the function $f \in \mathcal{K}$ are developed. Section 4.3 contains the study of the

dynamics of $f_\lambda \in \mathcal{K}$ on the real line. In this section, it is shown that bifurcation in the dynamics of $f_\lambda(x)$ occurs at $\lambda = \lambda^* (\approx 0.64761)$ where $\lambda^* = (x^*)^2/(e^{x^*} - 1)$ and x^* is the unique positive real root of the equation $e^x(2 - x) - 2 = 0$. That is, if the parameter value crosses the value λ^*, then a sudden dramatic change in the dynamics of $f_\lambda(x)$ occurs. In Section 4.4, the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda^*$ is studied. For this case, we prove two different characterizations for the Julia set of $f_\lambda(z)$. The first characterization gives the Julia set $\mathcal{J}(f_\lambda)$ for $0 < \lambda < \lambda^*$ as the closure of the set of escaping points; while the second characterization, describes it as the complement of the basin of attraction of an attracting real fixed point of $f_\lambda(z)$. Further. in this section, it is found that, under a certain condition, the Julia set of $f_\lambda(z)$, $0 < \lambda < \lambda^*$, is a nowhere dense subset of the right half plane. In Section 4.5, the dynamical behaviour of $f_\lambda(z)$ for $\lambda > \lambda^*$ is described. We prove that the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$ contains the entire real line. The characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of escaping points, analogous to the first characterization in Section 4.4 is obtained in this case also. In Section 4.6, the characterizations of the Julia set of $f_\lambda(z)$, obtained in Sections 4.4 and 4.5, are applied to computationally generate the pictures of the Julia set of $f_\lambda(z)$ for different values of λ. Finally, the results of our investigations on the dynamics of the non-critically finite entire function $f_\lambda \in \mathcal{K}$ are compared with those of Devaney [26, 31], Devaney and Durkin [33], Devaney and Krych [36], Devaney and Tangerman [37] and Misiurewicz [75] obtained for the dynamics of the critically finite entire functions $E_\lambda(z) = \lambda\, e^z$.

Chapter 5

In Chapter 5, the dynamics of the entire function $h_\lambda(z) = \lambda h(z)$ where λ is a non-zero real parameter and $h(z) = \sinh z/z$ is an even non-critically finite entire function arising as a limit function of the sequence of denominators of the approximants of the modified general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{-z^2/((2n)(2n+1))}{1+(z^2/(2n)(2n+1))}\right)$ is studied. Let $\mathcal{H} \equiv \{h_\lambda(z) = \lambda h(z) :$

$\lambda \in \mathbb{R} \setminus \{0\}\}$. Section 5.2 is devoted to the results on some of the basic properties of $h_\lambda \in \mathcal{H}$. In Section 5.3, the dynamics of $h_\lambda(x)$ for $x \in \mathbb{R}$ is described. In this section, it is shown that there exists a critical parameter value $\lambda^{**} > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$ occurs at $|\lambda| = \lambda^{**}(\approx 1.104)$. The critical parameter λ^{**} is given by $\lambda^{**} = (x^{**})^2/\sinh x^{**}$ and x^{**} is the unique positive real root of the equation $\tanh x = x/2$. The dynamics of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < |\lambda| < \lambda^{**}$ is studied in Section 5.4. For this case, two different characterizations for the Julia set of $h_\lambda(z)$ are obtained. The first characterization gives the Julia set $\mathcal{J}(h_\lambda)$ for $0 < |\lambda| < \lambda^{**}$ as the closure of the set of escaping points; while the second characterization, describes it as the complement of the basin of attraction of an attracting real fixed point of $h_\lambda(z)$. Further, in this section, it is found that, under a certain condition, the Julia set of $h_\lambda(z)$, $0 < |\lambda| < \lambda^{**}$ is a nowhere dense subset of the complex plane. In Section 5.5, the dynamical behaviour of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| > \lambda^{**}$ is described. We prove that the Julia set of $h_\lambda(z)$ for $|\lambda| > \lambda^{**}$ contains all the real points and the purely imaginary points of the complex plane. The characterization of the Julia set of $h_\lambda(z)$ as the closure of the set of escaping points is obtained in this case. In Section 5.6, the characterizations of the Julia set obtained in Sections 5.4 and 5.5, are applied to computationally generate the pictures of the Julia set of $h_\lambda(z)$ for various values of λ. Further, the results obtained in this chapter for the dynamics of $h_\lambda \in \mathcal{H}$ are compared with those of Devaney and Durkin [33] obtained for the dynamics of critically finite even entire function $C_\lambda(z) = \lambda i \cos z$, $\lambda \in \mathbb{R} \setminus \{0\}$ and, finally, a comparison is made in this section between the results on the dynamics of $f_\lambda \in \mathcal{K}$ and $h_\lambda \in \mathcal{H}$, $\lambda > 0$, as found in Chapter 4 and in the present chapter.

Chapter 6

In Chapter 6, a class of non-critically finite entire functions is introduced and it is proved that explosion occurs in the Julia sets of functions in one parameter family generated from

each function in this class. Let $\mathcal{F}$ be the class of functions $f(z)$ satisfying (i) $f(z)$ is an entire function having order ρ with $(1/2) \leq \rho < 1$, (ii) $f(z)$ has only negative real zeros in the complex plane, (iii) $|f(-x)| \leq f(0) = 1$, for all $x > 0$ and (iv) $\lim_{x\to\infty} f(-x) = 0$; and $\mathcal{G}$ be the class of functions defined by $\mathcal{G} = \{g(z) = f(z^2) : f \in \mathcal{F}\}$. In the present chapter, the dynamics of $g_\lambda(z) = \lambda g(z)$, $\lambda \in \mathsf{R} \setminus \{0\}$, for a function $g \in \mathcal{G}$, is studied. Let $\mathcal{S} \equiv \{g_\lambda(z) \ : \ \lambda \in \mathsf{R} \setminus \{0\}\}$. Section 6.2 describes the bifurcation in the dynamics of functions $g_\lambda \in \mathcal{S}$ for $x \in \mathsf{R}$. It is shown that there exists a critcal parameter $\lambda_g^* > 0$ such that bifurcation in the dynamics of functions in $\mathcal{S}$ for $x \in \mathsf{R}$ occurs at $|\lambda| = \lambda_g^*$. In Section 6.3, the dynamics of $g_\lambda \in \mathcal{S}$ for $z \in \mathsf{C}$ is described and the chaotic burst in the Julia sets of functions in the family $\mathcal{S}$ is exhibited. It is shown that the Fatou set of $g_\lambda(z)$ is an unbounded proper subset of the complex plane when $0 < |\lambda| \leq \lambda_g^*$ and consequently Julia set of $g_\lambda(z)$ is also unbounded proper subset of the complex plane for this case, while the Julia set of $g_\lambda(z)$ is the extended complex plane when $|\lambda| > \lambda_g^*$. Finally, certain interesting examples of the family $\mathcal{S}$, viz, (i) $\mathcal{I} \equiv \{\lambda I_0(z) \ : \ \lambda \in \mathsf{R} \setminus \{0\}\}$, where $I_0 \in \mathcal{G}$ is the well known modified Bessel function of zero order arising as the denominator of the separately convergent modified general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{-z^2/(2n)^2}{1 + z^2/(2n)^2} \right)$ and (ii) $\mathcal{M}_k \equiv \{\lambda G_{2k}(z) \ : \ G_{2k}(z) = F_{2k}(iz)/F_{2k}(0),\ \lambda \in \mathsf{R} \setminus \{0\}\}$, where $G_{2k} \in \mathcal{G}$ with fixed $k = 1, 2, \ldots$ and $F_{2k}(z) = \int_0^\infty e^{-t^{2k}} \cos zt \, dt$, are given and the picture of the Julia set of functions in the family $\mathcal{I}$ is computationally generated for various values of λ.

Chapter 2

Growth of Entire Functions Arising from Separately Convergent General T-Fractions

Let $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$ for $n \geq 1$, be a general T-fraction. If $\sum |F_n| < \infty$ and $\sum |G_n| < \infty$, the general T-fraction converges separately (c.f. Definition 1.2.2) to entire functions ([98], c.f. Theorem 1.2.2). Let $A(z)$ and $B(z)$ denote respectively the limit functions of the sequences of numerators and denominators of the approximants of the general T-fraction. It is known ([98], c.f. Theorem 1.3.5) that, for a general T-fraction satisfying $\sum |F_n| < \infty$ and $\sum |G_n| < \infty$, the entire functions $A(z)$ and $B(z)$ are of order (c.f. Definition 1.3.1) atmost one. In the present chapter, the growth of the entire functions $A(z)$ and $B(z)$ is studied by investigating the influence of the elements F_n and G_n of a general T-fraction on the generalized $(\alpha\,,\,\alpha)$- order (c.f. Definition 1.3.6) and generalized $(\alpha\,,\,\beta)$- order (c.f. Definition 1.3.4) of the entire functions $A(z)$ and $B(z)$. In Section 2.1, a sufficient condition on the elements of general T-fraction for $A(z)$ and $B(z)$ to be of slow growth (c.f. Section 1.3) is obtained that generalizes a result of Thron [96] for regular C-fractions. The results obtained in Sections 2.2, on generalized $(\alpha\,,\,\alpha)$- order of $A(z)$ and $B(z)$ are specially suited for the comparison of growth of the entire functions $A(z)$ and $B(z)$ when these are of order zero and extend some of the recent results in [64]. The results

pertaining to generalized (α, α)- order of $A(z)$ and $B(z)$ are used in Chapter 3 to study the dynamics of certain entire transcendental functions of slow growth. In Section 2.3, we obtain a sufficient condition on the elements of a general T-fraction for $A(z)$ and $B(z)$ to be of generalized lower (α, β)- order not less than a prespecified constant. This leads to a generalization of a result of Maillet [70] obtained for regular C-fractions. Further, in this section, the influence of the elements of a general T-fraction on the generalized (α, β)- order of $A(z)$ (or $B(z)$) is investigated. In section 2.4, in order to obtain even entire functions as numerators and denominators of continued fractions, a modified general T-fraction is introduced and the growth of the numerators and the denominators of modified general T-fractions, when they are of slow growth, is studied.

2.1 Elements of a general T-fraction and order of its numerator and denominator

Let

$$C(z) \equiv \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{F_n\, z}{1 + G_n\, z}\right), \qquad F_n \neq 0, \quad n \geq 1 \tag{2.1.1}$$

be a general T-fraction satisfying the condition

$$\sum_{n=1}^{\infty} |F_n| < \infty \qquad \sum_{n=1}^{\infty} |G_n| < \infty\,. \tag{2.1.2}$$

Let $A_n(z)$ and $B_n(z)$ denote respectively the numerator and the denominator of the nth approximant of the general T-fraction (2.1.1). By Theorem 1.2.2, the sequences $\{A_n(z)\}_{n=1}^{\infty}$ and $\{B_n(z)\}_{n=1}^{\infty}$ converge uniformly on compact subsets of C to entire functions $A(z)$ and $B(z)$ respectively. Thus, the condition (2.1.2) is sufficient for the separate convergence of the general T-fraction $C(z)$. The following example shows that the condition (2.1.2) is not necessary for the separate convergence of a general T-fraction.

Example 2.1.1. *Consider the general T-fraction*

$$\operatorname*{K}_{n=1}^{\infty}\left(\frac{-z/n}{1+(z/n)}\right) . \tag{2.1.3}$$

Let $B_n(z) = 1+\sum_{k=1}^{n} q_k^{(n)} z^k$ be the denominator of the nth approximant of (2.1.3). From the three term recurrence relation (1.2.2) for $B_n(z)$, it is easily seen that $q_k^{(n)} = \frac{1}{k!}$ for $1 \le k \le n$, and so

$$\lim_{n\to\infty} B_n(z) = \lim_{n\to\infty}\left(1+\sum_{k=1}^{n}\frac{z^k}{k!}\right) = 1+\sum_{k=1}^{\infty}\frac{z^k}{k!} = e^z \equiv B(z) .$$

Thus, the sequence $\{B_n(z)\}_{n=1}^{\infty}$ converges to the entire function $B(z) = e^z$ for all $z \in \mathbb{C}$. Since $A_n(z) = 1 - B_n(z)$, the convergence of $\{A_n(z)\}$ to $A(z) = 1 - e^z$ follows by the convergence of $\{B_n(z)\}$. Since $\sum |F_n| = \sum |G_n| = \sum \frac{1}{n} = \infty$, it follows that for the separate convergence of the general T-fraction the condition (2.1.2) is not necessary.

The following is the Śleszyński type inequality [90] for the denominator $B_n(z)$ of the nth approximant of a general T-fraction:

Proposition 2.1.1. *Let $C(z) \equiv \operatorname*{K}_{n=1}^{\infty}\left(\frac{F_n\, z}{1+G_n\, z}\right)$ be a general T-fraction and $B_n(z) = \sum_{k=0}^{n} q_k^{(n)} z^k$ be the denominator of its nth approximant. Then, for $n \ge 2k$,*

$$\left|q_k^{(n)}\right| \le \sum_{d=0}^{k}\left(\prod_{m=1}^{d}\sigma_{2m}^{(n)}\right)\left(\prod_{m=1}^{k-d}\omega_m^{(n)}\right) \tag{2.1.4}$$

where, $\prod_{m=1}^{i}\alpha_m \equiv 1$ for $i=0$, $\sigma_m^{(n)} = \sum_{k=m}^{n}|F_k|$ and $\omega_m^{(n)} = \sum_{k=m}^{n}|G_k|$.

Proof. Let $B_n(z) = \sum_{k=0}^{n} q_k^{(n)} z^k$. From (1.2.2), it is readily seen that $B_n(z)$ satisfies the three term recurrence relation

Equating the coefficients of z^k in the above equation,

$$q_k^{(n)} = q_k^{(n-1)} + G_n q_{k-1}^{(n-1)} + F_n q_{k-1}^{(n-2)}, \qquad 1 \leq k \leq n\,. \tag{2.1.5}$$

Obviously, $q_0^{(n)} = 1$ for $n \geq 0$, since $B_n(0) = 1$ for all $n \geq 0$. Intrinsically, $q_k^{(n)}$ consists of a sum of product terms (without any repeatation of terms) of the form $\prod_{i=1}^{k} a_i$, where $a_i \in \{G_1, \ldots, G_n, F_2, \ldots, F_n\}$ with $a_i \neq a_j$ for $i \neq j$. These product terms are called *q-terms of order k*. Thus, $q_k^{(n)}(G_1, \ldots, G_n, F_2, \ldots, F_n)$ is a polynomial in the variables $G_1, \ldots, G_n, F_2, \ldots, F_n$ with only non-negative coefficients. Define,

$$\widetilde{q_k}^{(n)}(G_1, \ldots, G_n, F_2, \ldots, F_n) = q_k^{(n)}(|G_1|, \ldots, |G_n|, |F_2|, \ldots, |F_n|)\,.$$

Clearly, $\left|q_k^{(n)}\right| \leq \widetilde{q_k}^{(n)}$.

Since each term of $\widetilde{q_k}^{(n)}$ is a q-term of order k and is one of the terms in $\sum_{d=0}^{k}\left(\prod_{m=1}^{d}\sigma_{2m}^{(n)}\right)\left(\prod_{m=1}^{k-d}\omega_m^{(n)}\right)$, it follows that

$$\widetilde{q_k}^{(n)} \leq \sum_{d=0}^{k}\left(\prod_{m=1}^{d}\sigma_{2m}^{(n)}\right)\left(\prod_{m=1}^{k-d}\omega_m^{(n)}\right).$$

Thus, in view of $\left|q_k^{(n)}\right| \leq \widetilde{q_k}^{(n)}$, the inequality (2.1.4) follows from the above inequality. □

Corollary 2.1.1. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\dfrac{F_n\, z}{1 + G_n\, z}\right)$ *be a general T-fraction and* $B_n(z) = \sum_{k=0}^{n} q_k^{(n)} z^k$ *be the denominator of its nth approximant. Then, for* $n \geq 2k$,

$$\left|q_k^{(n)}\right| \leq \sum_{d=0}^{k}\left(\sigma_2^{(n)}\right)^d\left(\omega_1^{(n)}\right)^{k-d} \tag{2.1.6}$$

where, $\sigma_m^{(n)} = \sum_{k=m}^{n}|F_k|$ *and* $\omega_m^{(n)} = \sum_{k=m}^{n}|G_k|$.

Proof. Using $\sigma_{2m}^{(n)} \leq \sigma_2^{(n)}$ and $\omega_m^{(n)} \leq \omega_1^{(n)}$ for $m \geq 1$, the inequality (2.1.6) follows from (2.1.4). □

Remark 2.1.1. *The Śleszyński type inequality for the numerator $A_n(z)$ of the nth approximant of a general T-fraction is readily obtained following the lines of the proof of Proposition* 2.1.1. *Thus, for a general T-fraction $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, if $A_n(z) = \sum_{k=0}^{n} p_k^{(n)} z^k$ is the numerator of its nth approximant, then*

$$\left|p_{k+1}^{(n)}\right| \leq |F_1| \sum_{d=0}^{k} \left(\prod_{m=1}^{d} \sigma_{2m}^{(n)}\right) \left(\prod_{m=1}^{k-d} \omega_m^{(n)}\right), \qquad n \geq 2k \tag{2.1.7}$$

where, $\prod_{m=1}^{i} \alpha_m \equiv 1$ *for* $i = 0$, $\sigma_m^{(n)} = \sum_{k=m}^{n} |F_k|$ *and* $\omega_m^{(n)} = \sum_{k=m}^{n} |G_k|$.

The following proposition establishes the convergence of the sequence $\{q_k^{(n)}\}_{n=1}^{\infty}$ for each positive integer k:

Proposition 2.1.2. *Let $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$ for $n \geq 1$, be a general T-fraction satisfying* (2.1.2)*. Let $B(z) = \lim_{n\to\infty} B_n(z)$ where $B_n(z) = \sum_{k=0}^{n} q_k^{(n)} z^k$ is the denominator of nth approximant of the general T-fraction and $B(z) = \sum_{k=0}^{n} q_k z^k$. Then, for each integer $k \geq 0$, the sequence $\left\{q_k^{(n)}\right\}_{n=1}^{\infty} \to q_k$ as $n \to \infty$.*

Proof. Let $B_n^{(k)}(z)$ denote the kth derivative of $B_n(z)$. It is known [98] that under the condition (2.1.2), $\{B_n(z)\}$ converges uniformly to the entire function $B(z) = \sum_{k=0}^{\infty} q_k z^k$. Thus, $B_n^{(k)}(z) \to B^{(k)}(z)$ as $n \to \infty$, for all $z \in C$, $k \geq 1$ ([100], p98). In particular, $B_n^{(k)}(1) \to B^{(k)}(1)$ as $n \to \infty$ for $k \geq 1$. Consequently, for $k \geq 1$, $q_k^{(n)} \to q_k$ as $n \to \infty$. Since $q_0^{(n)} = 1$, for all $n \geq 0$, $q_0^{(n)} \to q_0 = 1$ as $n \to \infty$. Therefore, $\lim_{n\to\infty} q_k^{(n)} = q_k$, $k \geq 0$. □

Remark 2.1.2. *We observe that Proposition* 2.1.2 *continues to hold if $B_n(z)$ and $B(z)$ are respectively replaced by the numerator $A_n(z)$ of nth approximant of $C(z)$ and the numerator $A(z)$ of $C(z)$. Thus, let $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$, $F_n \neq 0$ for $n \geq 1$, be a general T-fraction satisfying* (2.1.2) *and $A(z) = \lim_{n\to\infty} A_n(z)$, where $A_n(z) = \sum_{k=0}^{n} p_k^{(n)} z^k$, $A(z) = \sum_{k=0}^{n} p_k z^k$. Then, for each integer $k \geq 0$, the sequence $\left\{p_k^{(n)}\right\}_{n=1}^{\infty} \to p_k$ as $n \to \infty$.*

Since [98],

$$|B_n(z)| < \prod_{k=1}^{n} (1 + (|F_k| + |G_k|)|z|)$$

the inequaltiy $|B(z)| < \prod_{k=1}^{\infty} (1 + (|F_k| + |G_k|)|z|) < e^{|z|(\sum_{k=1}^{\infty}(|F_k|+|G_k|))}$ holds. Consequently, $\rho(B) = \limsup \frac{\log\log M(B,r)}{\log r} \leq 1$. Thus, all the entire functions $B(z)$ (and by an analogous argument $A(z)$) arising as the denominators of general T-fractions satisfying (2.1.2), are of order atmost one.

The following theorem gives a sufficient conditions on the elements of a general T-fraction (2.1.1) for its numerator and denominator to be of order zero.

Theorem 2.1.1. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1 + G_n\, z}\right)$, $F_n \neq 0$ *for* $n \geq 1$, *be a general T-fraction. If, for* $n \geq 1$, $|F_n| \leq r^n$ *and* $|G_n| \leq r^n$, $0 < r < 1$, *the numerator* $A(z)$ *and the denominator* $B(z)$ *of the continued fraction* $C(z)$ *are entire functions of order zero.*

Proof. We prove the theorem for $B(z)$ and for $A(z)$ it follows by analogous arguments. Let $B(z) = \sum_{k=0}^{\infty} q_k z^k$. For $n \geq 1$, the conditions $|F_n| \leq r^n$ and $|G_n| \leq r^n$, $0 < r < 1$. imply that the condition (2.1.2) is satisfied so that $B(z)$ is an entire function. Further, for each $m \geq 1$,

$$\sigma_m = \sum_{n=m}^{\infty} |F_n| \leq \sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r} \qquad \text{and} \qquad \omega_m = \sum_{n=m}^{\infty} |F_n| \leq \sum_{n=m}^{\infty} r^n = \frac{r^m}{1-r}.$$

Therefore,

$$\prod_{m=1}^{d} \sigma_m \leq \frac{r^{d(d+1)/2}}{(1-r)^d}$$

and

$$\prod_{m=1}^{k-d} \omega_m \leq \frac{r^{(k-d)(k-d+1)/2}}{(1-r)^{k-d}}.$$

Since $\lim_{n\to\infty} q_k^{(n)} = q_k$, it follows that

$$|q_k| \leq \sum_{d=0}^{k} \left(\prod_{m=1}^{d} \sigma_{2m}\right) \left(\prod_{m=1}^{k-d} \omega_m\right) \leq \sum_{d=0}^{k} \left(\prod_{m=1}^{d} \sigma_m\right) \left(\prod_{m=1}^{k-d} \omega_m\right) \tag{2.1.8}$$

Consequently,

$$\begin{aligned}
|q_k| &< \sum_{d=0}^{k} \left(\frac{r^{d(d+1)/2}}{(1-r)^d}\right) \left(\frac{r^{(k-d)(k-d+1)/2}}{(1-r)^{k-d}}\right) \\
&< \frac{1}{(1-r)^k} \sum_{d=0}^{k} \left(r^{d(d+1)/2}\right) \left(r^{(k-d)(k-d+1)/2}\right) \\
&< \frac{k+1}{(1-r)^k}\, M
\end{aligned} \tag{2.1.9}$$

where $M = \max_{0\le d\le k} \left\{ \left(r^{d(d+1)/2}\right) \left(r^{(k-d)(k-d+1)/2}\right) \right\}$. It is easily seen that

$$M = \begin{cases} r^{\frac{k}{2}+\frac{k^2}{4}} & \text{if } k \text{ is even} \\ r^{\frac{1}{4}+\frac{k}{2}+\frac{k^2}{4}} & \text{if } k \text{ is odd} \end{cases}$$

Therefore, by (2.1.9),

$$-\log|q_k| \ge -\log(k+1) + \frac{k^2}{4}\log\frac{1}{r} + \frac{k}{2}\log\frac{1}{r} + k\log(1-r) + a_k(r) \tag{2.1.10}$$

where $a_k(r) = 0$ if k is even and $a_k(r) = (-1/4)\log r$ if k is odd. Let $\rho \equiv \rho(B)$ be the order of $B(z)$. It follows by (1.3.1) and (2.1.10) that

$$\rho = \limsup_{k\to\infty} \frac{k\log k}{-\log|q_k|} = 0. \qquad \square$$

Remark 2.1.3. *Theorem* 2.1.1 *generalizes a result of Thron ([96], c.f. Theorem* 1.3.4*) obtained for a regular C-fraction* $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1}\right)$, *where* $F_n \neq 0$ *for* $n \ge 1$.

2.2 Generalized (α, α)- order of the numerator and the denominator of a general T-fraction

Theorem 1.3.4 only gives that if $|F_n| \le r^{-n}$, $0 < r < 1$, for a regular C-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1}\right)$ the orders of its numerator $A(z)$ and denominator $B(z)$ are zero. For such regular C-fractions, the order of their numerators (*or* denominators) can not be compared satisfactorily by confining to the above result of Thron. In the present section, an attempt to overcome this shortcoming in the result of Thron is made in a general set up by considering general T-fractions $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$ and finding the influence of F_n and G_n on

$x_0 \equiv x_0(c)$ such that the function

$$\mathsf{G}[x;\, c] = \alpha^{-1}(c\alpha(x)), \quad 0 < c < 1, \qquad x \in \mathsf{R}$$

is constant for $x \in (-\infty, x_0)$ and $\mathsf{G}[x;\, c]$ is strictly increasing for $x \in [x_0, \infty)$.

The following theorem gives sufficient conditions on the elements of a general T-fraction for its numerator and denominator to have generalized $(\alpha\,,\ \alpha)$- order equal to a prespecified constant $\mu \geq 1$.

Theorem 2.2.1. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{F_n\, z}{1 + G_n\, z}\right)$ *be a general T-fraction satisfying (2.1.2) and* $\rho(\alpha, \alpha; f)$ *and* $\lambda(\alpha, \alpha; f)$ *be the generalized* (α, α)*-order and the generalized lower* (α, α)*- order of* $f(z)$*, where* $f \equiv A$ *or* $f \equiv B$*;* $A(z)$ *and* $B(z)$ *being respectively the numerator and the denominator of the continued fraction* $C(z)$*. If* $\mathsf{G}'[x;\, \frac{1}{\mu}] \equiv \frac{d}{dx}\left(\mathsf{G}[x;\, \frac{1}{\mu}]\right)$ *monotonically decreases to zero as* $x \to \infty$*, where* $\mathsf{G}[x;\, \frac{1}{\mu}] = \alpha^{-1}(\frac{1}{\mu}\alpha(x))$*,* $\alpha(x) \in \Omega$*,* $1 \leq \mu < \infty$*; and*

(i) if, for $n \geq 1$*,* $|F_n| \leq \dfrac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]}$ *and* $|G_n| \leq \dfrac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]}$*, then* $\rho(\alpha, \alpha; f) \leq \mu$*.*

(ii) if, for $n \geq 1$*,* $F_n \geq \dfrac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]}$ *and* $G_n \geq \dfrac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]}$*, then* $\lambda(\alpha, \alpha; f) \geq \mu$*.*

Proof. We prove the theorem for $f \equiv B$; for $f \equiv A$ it follows by analogous arguments.

(i) Let $\quad B(z) = \sum_{k=0}^{\infty} q_k\, z^k$.

The conditions $|F_n| \leq \dfrac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]}$ and $|G_n| \leq \dfrac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]}$, $n \geq 1$, imply that

$$\begin{aligned} \sigma_m \;&=\; \sum_{n=m}^{\infty} |F_n| \leq \sum_{n=m}^{\infty} \frac{1}{n^2 \exp \mathsf{G}[n\,;\, \frac{1}{\mu}]} \\ &\leq \int_{m-1}^{\infty} \frac{dx}{x^2 \exp \mathsf{G}[x\,;\, \frac{1}{\mu}]} \leq \frac{1}{\exp \mathsf{G}[m-1\,;\, \frac{1}{\mu}]} \qquad \text{for } m \geq 2 \end{aligned}$$

and

$$\sigma_1 \;\leq\; |F_1| + \sum_{n=2}^{\infty} |F_n| \leq \frac{2}{\exp\left(\mathsf{G}[1\,;\, \frac{1}{\mu}]\right)} \,.$$

Similarly,

$$\omega_m \;\leq\; \frac{1}{\exp \mathsf{G}[m-1\,;\,\frac{1}{\mu}]} \qquad \text{for } m \geq 2$$

and

$$\omega_1 \;\leq\; |G_1| + \sum_{n=2}^{\infty} |G_n| \leq \frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}]\right)}\,.$$

From (2.1.8) and the above inequalities, it follows that

$$\begin{aligned} |q_k| \;&\leq\; \sum_{d=0}^{k} \left(\frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[d-1\,;\,\frac{1}{\mu}]\right)} \right) \times \\ &\quad \left(\frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[k-d-1\,;\,\frac{1}{\mu}]\right)} \right) \qquad (2.2. \end{aligned}$$

<u>Case 1: k is even</u>

By (2.2.1),

$$\begin{aligned} |q_k| \;&\leq\; 2\sum_{d=0}^{k/2} \left(\frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[d-1\,;\,\frac{1}{\mu}]\right)} \right) \times \\ &\quad \left(\frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[k-d-1\,;\,\frac{1}{\mu}]\right)} \right) \\ &\leq\; 2\sum_{d=0}^{k/2} \left(\frac{4}{\exp\left(2\{2\;\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[(k/2)-1\,;\,\frac{1}{\mu}]\}\right)} \right) \\ &\leq\; (k+2)\,\frac{4}{\exp\left(2\{2\;\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[(k/2)-1\,;\,\frac{1}{\mu}]\}\right)} \\ &\leq\; (k+2)\,\frac{4}{\exp\left(2\{\,\mathsf{G}[1\,;\,\frac{1}{\mu}] + \sum_{m=1}^{(k/2)-1} \mathsf{G}[m\,;\,\frac{1}{\mu}]\}\right)} \end{aligned}$$

Let j be the greatest integer such that $\mathsf{G}[1\,;\,\frac{1}{\mu}] = \mathsf{G}[2\,;\,\frac{1}{\mu}] = \cdots = \mathsf{G}[j-1\,;\,\frac{1}{\mu}] = \mathsf{G}[j\,;$ and $\mathsf{G}[j\,;\,\frac{1}{\mu}] \neq \mathsf{G}[j+1\,;\,\frac{1}{\mu}]$. Since $\mathsf{G}[x;\frac{1}{\mu}]$, $0 < \frac{1}{\mu} < 1$ is increasing and $\frac{d}{dx}\mathsf{G}[x;\frac{1}{\mu}]$, $0 < \frac{1}{\mu} <$

is monotonically decreasing for $x \geq x_0$, it follows by Lemma 2.2.1 that, for $k > j$,

$$\begin{aligned}
\sum_{m=1}^{(k/2)-1} \mathsf{G}[m\,;\,\tfrac{1}{\mu}] &\geq (j-1)\,\mathsf{G}[j\,;\,\tfrac{1}{\mu}] + \sum_{m=j}^{(k/2)-1} \mathsf{G}[m\,;\,\tfrac{1}{\mu}] \\
&\geq (j-1)\,\mathsf{G}[j\,;\,\tfrac{1}{\mu}] + \left(\frac{\frac{k}{2}-j+1}{2}\right) \mathsf{G}[j\,;\,\tfrac{1}{\mu}] + \left(\frac{\frac{k}{2}-j-1}{2}\right) \mathsf{G}[\tfrac{k}{2}-1\,;\,\tfrac{1}{\mu}] \\
&\geq \left(\frac{\frac{k}{2}+j-1}{2}\right) \mathsf{G}[j\,;\,\tfrac{1}{\mu}] + \left(\frac{\frac{k}{2}-j-1}{2}\right) \mathsf{G}[\tfrac{k}{2}-1\,;\,\tfrac{1}{\mu}]
\end{aligned}$$

Thus,

$$\log|q_k|^{-1} \geq -\log(4k+8) + 2\left\{\left(\frac{\frac{k}{2}+j+1}{2}\right) \mathsf{G}[j\,;\,\tfrac{1}{\mu}] + \left(\frac{\frac{k}{2}-j-1}{2}\right) \mathsf{G}[\tfrac{k}{2}-1\,;\,\tfrac{1}{\mu}]\right\} \quad (2.2.$$

Since $\frac{d}{dx}\mathsf{G}[x;\frac{1}{\mu}]$, $0 < \frac{1}{\mu} < 1$ is monotonically decreasing to zero as $x \to \infty$, it follows th $\phi(x) = \frac{\mathsf{G}[x;\frac{1}{\mu}]}{x}$, $0 < \frac{1}{\mu} < 1$ tends to zero as $x \to \infty$. Therefore,

$$\frac{1}{k}\log|q_k|^{-1} \geq O\left(\mathsf{G}[\tfrac{k}{2}-1\,;\,\tfrac{1}{\mu}]\right) \qquad \text{as} \quad k \to \infty\,. \qquad (2.2.$$

<u>Case 2: k is odd</u>

By (2.2.1),

$$\begin{aligned}
|q_k| \leq\ & 2\sum_{d=0}^{(k-1)/2} \left(\frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[d-1\,;\,\frac{1}{\mu}]\right)}\right) \times \\
& \left(\frac{2}{\exp\left(\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[k-d-1\,;\,\frac{1}{\mu}]\right)}\right) \\
\leq\ & 2\sum_{d=0}^{(k-1)/2} \left(\frac{4}{\exp\left(2\{2\,\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[\frac{k-1}{2}\,;\,\frac{1}{\mu}]\} + \mathsf{G}[\frac{k+1}{2}\,;\,\frac{1}{\mu}]\right)}\right) \\
\leq\ & (k+1)\,\frac{4}{\exp\left(2\{2\,\mathsf{G}[1\,;\,\frac{1}{\mu}] + \mathsf{G}[2\,;\,\frac{1}{\mu}] + \cdots + \mathsf{G}[\frac{k-1}{2}\,;\,\frac{1}{\mu}]\} + \mathsf{G}[\frac{k+1}{2}\,;\,\frac{1}{\mu}]\right)} \\
\leq\ & (k+1)\,\frac{4}{\exp\left(2\{\,\mathsf{G}[1\,;\,\frac{1}{\mu}] + \sum_{m=1}^{(k-1)/2} \mathsf{G}[m\,;\,\frac{1}{\mu}]\} + \mathsf{G}[\frac{k+1}{2}\,;\,\frac{1}{\mu}]\right)}
\end{aligned}$$

Let j be as in Case 1. Since, $\mathsf{G}[x;\frac{1}{\mu}]$, $0 < \frac{1}{\mu} < 1$ is increasing and $\frac{d}{dx}\mathsf{G}[x;\frac{1}{\mu}]$, $0 < \frac{1}{\mu} < 1$

monotonically decreasing for $x \geq x_0$, it follows by Lemma 2.2.1 that, for $k > j$,

$$\begin{aligned}
\sum_{m=1}^{(k-1)/2} \mathsf{G}[m;\tfrac{1}{\mu}] &\geq (j-1)\,\mathsf{G}[j;\tfrac{1}{\mu}] + \sum_{m=j}^{(k-1)/2} \mathsf{G}[m;\tfrac{1}{\mu}] \\
&\geq (j-1)\,\mathsf{G}[j;\tfrac{1}{\mu}] + \left(\frac{\frac{k-1}{2}-j+2}{2}\right)\mathsf{G}[j;\tfrac{1}{\mu}] + \left(\frac{\frac{k-1}{2}-j}{2}\right)\mathsf{G}[\tfrac{k-1}{2};\tfrac{1}{\mu}] \\
&\geq \left(\frac{k+2j-1}{4}\right)\mathsf{G}[j;\tfrac{1}{\mu}] + \left(\frac{k-2j-1}{4}\right)\mathsf{G}[\tfrac{k-1}{2};\tfrac{1}{\mu}]
\end{aligned}$$

Thus,

$$\begin{aligned}
\log|q_k|^{-1} \;\geq\; &-\log(4k+4) + 2\left\{\left(\frac{k+2j+3}{4}\right)\mathsf{G}[j;\tfrac{1}{\mu}] + \left(\frac{k-2j-1}{4}\right)\mathsf{G}[\tfrac{k-1}{2};\tfrac{1}{\mu}]\right\} \\
&+\mathsf{G}[\tfrac{k+1}{2};\tfrac{1}{\mu}] \qquad (2.2.4)
\end{aligned}$$

Since $\frac{d}{dx}\mathsf{G}[x;\frac{1}{\mu}]$, $0 < \frac{1}{\mu} < 1$ is monotonically decreasing to zero as $x \to \infty$, it follows by the inequality (2.2.4) that, as $k \to \infty$,

$$\frac{1}{k}\log|q_k|^{-1} \geq O\left(\mathsf{G}[\tfrac{k-1}{2};\tfrac{1}{\mu}]\right) . \qquad (2.2.5)$$

Since, $\alpha(x) \in \Omega$, (2.2.3), (2.2.5) and (1.3.11) give,

$$\tilde{L} \equiv \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha\left(\frac{1}{k}\log|q_k|^{-1}\right)} \leq \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha\left(A\ \mathsf{G}[k;\frac{1}{\mu}]\right)} \leq \mu .$$

and so, by (1.3.10)

$$\rho(\alpha,\alpha;B) \leq \mu .$$

(ii) Letting $n = 2k$ in (2.1.5),

$$q_k^{(2k)} = q_k^{(2k-1)} + G_{2k}q_{k-1}^{(2k-1)} + F_{2k}q_{k-1}^{(2(k-1))} .$$

Since, $q_k^{(n)} > 0$ for all n and k, and $G_n > 0$ for all $n > 0$, by the above equation $q_k^{(2k)} > F_{2k}q_{k-1}^{(2(k-1))}$. Therefore, since $q_k^{(n)}$ increases with n, it follows that $q_k^{(n)} > q_k^{(2k)} > F_2F_4\cdots F_{2(k-1)}F_{2k}$ for all $n > 2k$. Thus,

$$q_k \geq F_2\,F_4\cdots F_{2(k-1)}\,F_{2k} \qquad (2.2.6)$$

By (2.2.6) and the condition $F_n \geq \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$, $n \geq 1$,

$$q_k \;\geq\; \left(\frac{1}{2k}\right)^{2k} \frac{1}{\exp\left(k \;\; \mathsf{G}[2k\,;\,\frac{1}{\mu}]\right)} .$$

Consequently, for all positive integer k,

$$\frac{1}{k}\log|q_k|^{-1} \;\leq\; 2\,\log 2k + \mathsf{G}[2k\,;\,\tfrac{1}{\mu}] .$$

Proceeding as in (i), it follows from the above inequality, (1.3.13) and (1.3.14) that

$$\lambda(\alpha,\alpha;B) \;\geq\; \mu. \qquad \square$$

Remark 2.2.1. *(i) The method of proof of Theorem* 2.2.1 *can be easily adopted to prove that if F_n and G_n satisfy weaker conditions $|F_n| \leq e^{-n}$, $|G_n| \leq e^{-n}$, $n \geq 1$, than those in Theorem* 2.2.1 *with $\mu = 1$, both the functions $A(z)$ and $B(z)$ are of generalized $(\alpha\,,\,\alpha)$- order atmost one. Since the generalized $(\alpha\,,\,\alpha)$- order of an entire function is always greater than or equal to one, it follows that both the functions $A(z)$ and $B(z)$ are of generalized $(\alpha\,,\,\alpha)$- order exactly equal to one.*
(ii) If $F_n = \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$ and $G_n = \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$ for $n = 1, 2, \cdots$, then it follows by Theorem 2.2.1 *that $B(z)$ (or $A(z)$) is of generalized regular (α,α)- growth i.e. $\rho(\alpha,\alpha,B) = \lambda(\alpha,\alpha,B) = \mu$.*

Example 2.2.1. *Let $\log^{[1]}(x) = \log(x)$ and $\log^{[p]}(x) = \log\left(\log^{[p-1]}(x)\right)$, $p = 2, 3, \cdots$; $\exp^{[1]}(x) = \exp(x)$ and $\exp^{[p]}(x) = \exp\left(\exp^{[p-1]}(x)\right)$, $p = 2, 3, \cdots$. Consider the regular C-fraction*

$$C_\mu(z) \equiv \mathop{\mathbf{K}}_{n=1}^{\infty} \left(\frac{z \,/\, n^2 \exp^{[p+1]}\left(\frac{1}{\mu}\log^{[p]}(n)\right)}{1}\right), \quad \mu > 1, \quad p > 1 \tag{2.2.7}$$

Since all the functions of the form $\delta(\log x)$, where $\delta(x) \in \Lambda$, are in Ω, the functions $\log^{[p]}(x)$, $p > 1$, are in Ω. By Theorem 2.2.1 *it follows that the generalized $(\log^{[p]}, \log^{[p]})$ - order of the denominator (or numerator) of the regular C-fraction $C_\mu(z)$ is equal to μ.*

Remark 2.2.2. *In Example 2.2.1, observe that by using Thron's result ([96], c.f. Theorem 1.3.4) the order of the denominator (or numerator) of $C_\mu(z)$ is 0 for each μ. Thus, by confining to Thron's result it is not possible to compare the growth of denominators (or numerators) of regular C-fractions $C_\mu(z)$ for different values of μ; whereas, Theorem 2.2.1 does provide distinct measures of growth for denominators (or numerators) of $C_\mu(z)$ for different values of μ.*

Next, we extend Theorem 2.2.1 to the case when $\alpha(x) \in \overline{\Omega}$. It is to be observed [62] that Ω and $\overline{\Omega}$ are disjoint classes contained in Λ (c.f. Section 1.3).

Theorem 2.2.2. *Let $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n z}{1+G_n z}\right)$ be a general T-fraction satisfying (2.1.2) and $\rho(\alpha,\alpha;f)$ and $\lambda(\alpha,\alpha;f)$ be the generalized (α,α)-order and the generalized lower (α,α)- order of $f(z)$, where $f \equiv A$ or $f \equiv B$; $A(z)$ and $B(z)$ being respectively the numerator and the denominator of the continued fraction $C(z)$. If $\mathsf{G}'[x;\frac{1}{\mu}] \equiv \frac{d}{dx}\left(\mathsf{G}[x;\frac{1}{\mu}]\right)$ monotonically decreases to zero as $x \to \infty$, where $\mathsf{G}[x;\frac{1}{\mu}] = \alpha^{-1}(\frac{1}{\mu}\alpha(x))$, $\alpha(x) \in \overline{\Omega}$, $1 \le \mu < \infty$; and*

(i) if, for $n \ge 1$, $|F_n| \le \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$ and $|G_n| \le \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$, then $\rho(\alpha,\alpha;f) \le 1+\mu$.

(ii) if, for $n \ge 1$, $F_n \ge \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$ and $G_n \ge \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$, then $\lambda(\alpha,\alpha;f) \ge 1+\mu$.

Proof. (i) Let $f(z) = B(z) = \sum_{k=0}^{\infty} q_k\, z^k$. Using the condition $|F_n| \le \frac{1}{n^2 \exp \mathsf{G}[n\,:\,\frac{1}{\mu}]}$ and $|G_n| \le \frac{1}{n^2 \exp \mathsf{G}[n\,;\,\frac{1}{\mu}]}$, $n \ge 1$, and proceeding on the same lines of proof as those of (i) of previous theorem, we arrive at the inequalities (2.2.3) and (2.2.5). Since $\alpha(x) \in \overline{\Omega}$, (2.2.3) and (2.2.5) imply that

$$\frac{1}{k}\log|q_k|^{-1} \;\ge\; O\left(\mathsf{G}[k\,;\,\tfrac{1}{\mu}]\right) \qquad \text{as} \quad k \to \infty\,.$$

Using the above inequality, and the condition $\alpha(x) \in \overline{\Omega}$, it follows that

$$\widetilde{L} \le \mu\,.$$

Consequently, by (1.3.10) and (1.3.11)

$$\rho(\alpha, \alpha; B) \leq 1 + \mu .$$

(ii) If $F_n \geq \frac{1}{n^2 \exp G[n;\, \frac{1}{\mu}]}$, $n \geq 1$, then as in case (ii) of Theorem 2.2.1

$$q_k \geq \left(\frac{1}{2k}\right)^{2k} \frac{1}{\exp k\, G[2k;\, \frac{1}{\mu}]} .$$

Thus,

$$\frac{1}{k} \log |q_k|^{-1} \leq 2 \log 2k + G[2k;\, \tfrac{1}{\mu}] .$$

Using the condition $\alpha(x) \in \overline{\Omega}$, it follows in view of (1.3.13) and (1.3.14) that $\lambda(\alpha, \alpha; B) \geq 1 + \mu$. For $f \equiv A$, (i) and (ii) follow by analogous arguments. □

Remark 2.2.3. *If $F_n = \frac{1}{n^2 \exp G[n;\, \frac{1}{\mu}]}$ and $G_n = \frac{1}{n^2 \exp G[n;\, \frac{1}{\mu}]}$ for $n = 1, 2, \cdots$ in Theorem 2.2.2, it follows that $B(z)$ (or $A(z)$) is of generalized regular (α, α)- growth i.e. $\rho(\alpha, \alpha, B) = \lambda(\alpha, \alpha, B) = 1 + \mu$.*

Example 2.2.2. *Consider the regular C-fraction*

$$C_\mu(z) \equiv \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{z \,/\, n^2 \exp\left(n^{\frac{1}{\mu}}\right)}{1} \right), \qquad \mu > 1 \tag{2.2.8}$$

By Theorem 2.2.2 it follows that the generalized (log, log)- *order of the numerator $A(z)$ and the denominator $B(z)$ of (2.2.8) is equal to $1 + \mu$.*

Remark 2.2.4. *In the above example, using Thron's result for regular C-fractions ([96], c.f. Theorem 1.3.4), for each $\mu > 1$, the numerator $A(z)$ and the denominator $B(z)$ of the continued fraction $C_\mu(z)$ of the form (2.2.8) are entire functions of order zero so that by confining to Thron's result it is not possible to compare the growth of the numerators (or denominators) of the continued fractions $C_\mu(z)$ for different values of μ, whereas, by Theorem 2.2.2 the generalized* (log, log)- *order of the numerator $A(z)$ and the denominator $B(z)$ of $C_\mu(z)$ is equal to $1 + \mu$, facilitating an easy comparison of the growth of numerators and denominators of (2.2.8) for different choices of μ.*

Let $A(x)$ and $B(x)$ be the restrictions on the real line of the numerator $A(z)$ and the denominator $B(z)$ of a separately convergent general T-fraction. Define, for $n = 0, 1, 2, \cdots$,

$$E_n(A) = \inf_{Q\in\pi_n} ||A - Q|| \qquad \text{and} \qquad E_n(B) = \inf_{Q\in\pi_n} ||B - Q|| \tag{2.2.9}$$

where, for $f \equiv A$ or B, $||f-Q|| = \sup_{-1\le x\le 1} |f(x)-Q(x)|$ and π_n denotes the set of polynomials of degree atmost n. The following theorem leads to an upper bound on the error of approximations $E_n(A)$ and $E_n(B)$.

Theorem 2.2.3. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{F_n\, z}{1+G_n\, z}\right)$ *be a general T-fraction satisfying* (2.1.2). *Let* $E_n(A)$ *and* $E_n(B)$ *be defined by* (2.2.9) *and* $L_f = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)}$, *where* $f \equiv A$ *or* $f \equiv B$; $A(z)$ *and* $B(z)$ *being respectively the numerator and the denominator of the continued fraction* $C(z)$. *If* $\mathsf{G}'[x;\, \frac{1}{\mu}] \equiv \frac{d}{dx}\left(\mathsf{G}[x;\, \frac{1}{\mu}]\right)$ *monotonically decreases to zero as* $x \to \infty$, *where* $\mathsf{G}[x;\, \frac{1}{\mu}] = \alpha^{-1}(\frac{1}{\mu}\alpha(x))$; $\alpha(x) \in \Omega$ *or* $\overline{\Omega}$, $1 \le \mu < \infty$; *and*

(i) if, for $n \ge 1$, $|F_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $|G_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, *then* $P(L_f) \le \mu$ *for* $\alpha(x) \in \Omega$ *and* $P(L_f) \le 1+\mu$ *for* $\alpha(x) \in \overline{\Omega}$.

(ii) if, for $n \ge 1$, $F_n \ge \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $G_n \ge \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, *then* $P(L_f) \ge \mu$ *for* $\alpha(x) \in \Omega$ *and* $P(L_f) \ge 1+\mu$ *for* $\alpha(x) \in \overline{\Omega}$.

Here, $P(\Psi) = \max\{1, \Psi\}$ *if* $\alpha(x) \in \Omega$, $P(\Psi) = 1 + \Psi$ *if* $\alpha(x) \in \overline{\Omega}$.

Proof. In view of Theorem 2.2.1(i), Theorem 2.2.2(i) and Theorem 1.3.2, it follows that $P(L_A)$ and $P(L_B)$ are atmost μ if $\alpha(x) \in \Omega$ and atmost $1+\mu$ if $\alpha(x) \in \overline{\Omega}$. Similarly, by Theorem 2.2.1(ii), Theorem 2.2.2(ii) and Theorem 1.3.2, (ii) follows. □

Remark 2.2.5. *It is easily seen by Theorem* 2.2.3 *that under the conditions (i) and (ii) the error in the polynomial approximation* $E_n(A)$ *(or* $E_n(B)$*) of the numerator* $A(z)$ *(or the denominator* $B(z)$*) of a general T-fraction is bounded above by*

$$E_n(A) \;\le\; \exp\left(-n\,\alpha^{-1}\left(\frac{1}{\mu+\varepsilon}\alpha(n)\right)\right) \qquad \text{if } \alpha(x) \in \Omega \text{ or } \overline{\Omega}$$

for $n \ge n_0 \equiv n_0(\varepsilon)$ *and* $\varepsilon > 0$.

Proof. We prove the theorem for $f \equiv B$. The proof for $f \equiv A$ is analogous.

Let $B(z) = \sum_{k=0}^{\infty} q_k \ z^k$. Since, by (2.2.6), $q_k \geq F_2 F_4 \cdots F_{2(k-1)} F_{2k}$ *for* $k \geq 1$, the condition $F_n \geq \frac{1}{\exp \mathsf{G}[n;\,\nu]}$, $n \geq 1$, implies that

$$q_k \geq \frac{1}{\exp\left(k \ \mathsf{G}[2k;\,\nu]\right)}$$

$$\frac{1}{k} \log |q_k|^{-1} \leq \mathsf{G}[2k;\,\nu] \tag{2.3.2}$$

In view of (1.3.7), the inequality (2.3.2) gives that $\lambda(\alpha, \beta; B) \geq (1/\nu)$. □

Remark 2.3.1. *Since* $\rho(\alpha, \beta; f) \geq \lambda(\alpha, \beta; f)$, $f \equiv A$ *or* B, *with* $\alpha(x) = \log x$, $\beta(x) = x$ *and* $G_n = 0$ *for all* n, *a result of Maillet [70] (c.f. Theorem* 1.3.3*) for a regular C-fraction follows from Theorem* 2.3.1

Example 2.3.1. *Consider the regular C-fraction*

$$C_{\nu,p}(z) \equiv \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{z/\exp\left(\log^{[p-1]}\left(\nu \log^{[p]}(n)\right)\right)}{1} \right), \qquad 1 < \nu < \infty, \quad p \geq 1 \tag{2.3.3}$$

For $p = 1$, *Maillet's result ([70], c.f. Theorem* 1.3.3*) gives that the (usual) order (c.f. Section 1.3) of the denominator (or the numerator) of the regular C-fraction* $C_{\nu,1}(z)$ *is atleast* $(1/\nu)$, $1 < \nu < \infty$, *while, by Theorem* 2.3.1, *it follows that, for every* $p \geq 1$, *the generalized* $(\log^{[p]}, \log^{[p-1]})$*-order of the denominator (or the numerator) of the regular C-fraction* $C_{\nu,p}(z)$ *is atleast* $(1/\nu)$, $1 < \nu < \infty$.

The following corollary leads to a lower bound on the error in the polynomial approximations $E_n(A)$ and $E_n(B)$ (c.f. (2.2.9)) of the numerator $A(z)$ and the denominator $B(z)$ of a general T-fraction $C(z)$:

Corollary 2.3.1. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{F_n z}{1 + G_n z} \right)$ *be a general T-fraction satisfying (*2.1.2*). Let* $E_n(A)$ *and* $E_n(B)$ *be defined by (*2.2.9*) and* $L_f^* = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)}$ *where* $f \equiv A$

or $f \equiv B$*;* $A(z)$ *and* $B(z)$ *being respectively the numerator and the denominator of the continued fraction* $C(z)$*. If* σ *is defined by* (2.3.1)*,* $\mathsf{G}[x;\, \nu] = \beta^{-1}(\nu\alpha(x))$, $0 \leq \frac{1}{\sigma} < \nu < \infty$ *where* $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$ *satisfy*

(i) $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x} = \mathcal{O}(1)$, $(0 < c < \infty)$, *as* $x \to \infty$

(ii) $\frac{\beta(x\psi(x))}{\beta(e^x)} \to 0$ *as* $x \to \infty$*, for some function* $\psi(x)$ *tending to* ∞ *as* $x \to \infty$

and, if, for $n \geq 1$, $F_n \geq \frac{1}{\exp \mathsf{G}[n\,;\, \nu]}$ *and* $G_n \geq \frac{1}{\exp \mathsf{G}[n\,;\, \nu]}$*, then* $L_f^* \geq \frac{1}{\nu}$.

Proof. Combining Theorem 2.3.1 and Theorem 1.3.1 and using the fact that $\lambda(\alpha, \beta; f) \leq \rho(\alpha, \beta; f)$, the proof of Corollary 2.3.1 follows easily. □

Remark 2.3.2. *It follows that under the conditions of Corollary* 2.3.1*, the error in the polynomial approximation* $E_n(A)$ *(or* $E_n(B)$*) of the numerator* $A(z)$ *(or the denominator* $B(z)$*) of a general T-fraction is bounded below by*

$$E_n(A) \geq \frac{1}{\exp\left(n\,\beta^{-1}((\nu - \varepsilon)\,\alpha(n))\right)}$$

for $n \geq n_0 \equiv n_0(\varepsilon)$ *and* $\varepsilon > 0$.

The following theorem gives a sufficient condition on the elements of a general T-fraction for its numerator and denominator to have generalized $(\alpha\,, \beta)$- order atmost a prespecified constant.

Theorem 2.3.2. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1 + G_n\, z}\right)$ *be a general T-fraction satisfying* (2.1.2) *and* $\rho(\alpha, \beta; f)$ *be the generalized* (α, β)*- order of* $f(z)$*, where* $f \equiv A$ *or* $f \equiv B$*;* $A(z)$ *and* $B(z)$ *being respectively the numerator and the denominator of the continued fraction* $C(z)$*. If* $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$ *are such that*

(i) $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x} = \mathcal{O}(1)$, $(0 < c < \infty)$, *as* $x \to \infty$

(ii) $\mathsf{G}'[x\,;\, \nu] = \frac{d}{dx}(\mathsf{G}[x;\, \nu])$ *is monotonically decreasing for* $x > x_0$ *where,* $\mathsf{G}[x\,;\, \nu] = \beta^{-1}(\nu\,\alpha\,(x))$, $0 \leq \frac{1}{\sigma} < \nu < \infty$ *and* σ *is defined by* (2.3.1)

(iii) there exist $x_0(c)$, such that $\beta(cx) \geq c\,\beta(x)$, for all $x > x_0(c)$

and. if, for $n \geq j$, $|F_n| \leq \dfrac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]}$ and $|G_n| \leq \dfrac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]}$, then $\rho(\alpha, \beta; f) \leq \frac{2}{\nu}$.

Proof. We prove the theorem for $f \equiv B$. The proof for $f \equiv A$ is analogous.

Let $\quad B(z) = \sum_{k=0}^{\infty} q_k\, z^k$.

The conditions $|F_n| \leq \dfrac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]}$ and $|G_n| \leq \dfrac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]}$, $n \geq j$, imply that, for $m \geq j+1$,

$$\begin{aligned}\sigma_m &= \sum_{n=m}^{\infty} |F_n| \leq \sum_{n=m}^{\infty} \frac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]} \\ &\leq \int_{m-1}^{\infty} (-1)\frac{d}{dx}\left(\frac{1}{\exp \mathsf{G}[x\,;\,\nu]}\right) = \frac{1}{\exp \mathsf{G}[m-1\,;\,\nu]}\end{aligned}$$

Further, since, for $1 < m \leq j$,

$$\sum_{n=1}^{j} |F_n| \leq M_1(j) < \infty$$

for some constant $M_1(j)$, it follows that, for $1 \leq m \leq j$,

$$\sigma_m = \sum_{n=m}^{j} |F_n| + \sum_{n=j+1}^{\infty} |F_n| \leq M_1(j) + \frac{1}{\exp \mathsf{G}[j\,;\,\nu]} \leq \frac{M_2(j)}{\exp \mathsf{G}[j\,;\,\nu]} .$$

where $M_2(j) = 1 + M_1(j) \exp \mathsf{G}[j\,;\,\nu]$. Similarly

$$\begin{aligned}\omega_m &\leq \frac{1}{\exp \mathsf{G}[m-1\,;\,\nu]} \qquad \text{for} \quad m \geq j+1 \\ \omega_m &\leq \frac{M_3(j)}{\exp \mathsf{G}[j\,;\,\nu]} \qquad \text{for} \quad 1 \leq m \leq j\end{aligned}$$

where $M_3(j) = 1 + \left(\sum_{n=1}^{j} |G_n|\right) \exp \mathsf{G}[j\,;\,\nu]$. From (2.1.8) and the above inequalities, it follows that for $k > j$,

$$\begin{aligned}|q_k| \leq\ & \sum_{d=0}^{k} \left(\frac{M_2^j}{\exp\left(\mathsf{G}[j\,;\,\nu] + \cdots + \mathsf{G}[j\,;\,\nu] + \mathsf{G}[j+1\,;\,\nu] + \cdots + \mathsf{G}[d-1\,;\,\nu]\right)}\right) \times \\ & \left(\frac{M_3^j}{\exp\left(\mathsf{G}[j\,;\,\nu] + \cdots + \mathsf{G}[j\,;\,\nu] + \mathsf{G}[j+1\,;\,\nu] + \cdots + \mathsf{G}[k-d-1\,;\,\nu]\right)}\right)\end{aligned}$$

Now, proceeding on similar lines of proof as of Theorem 2.2.1, we get that, if k is even,

$$\begin{aligned}\log|q_k|^{-1} \;&\geq\; -\log\left((k+2)M_2^j\,M_3^j\right) + 2\left\{\left(\frac{\frac{k}{2}+j-1}{2}\right)\mathsf{G}[j\,;\,\nu] + \left(\frac{\frac{k}{2}-j-1}{2}\right)\mathsf{G}[\tfrac{k}{2}-1\,;\,\nu]\right\}\\ &\quad +2(j-1)\,\mathsf{G}[j\,;\,\nu]\end{aligned}$$

and if k is odd,

$$\begin{aligned}\log|q_k|^{-1} \;&\geq\; -\log\left((k+1)M_2^j\,M_3^j\right) + 2\left\{\left(\frac{k+2j-1}{4}\right)\mathsf{G}[j\,;\,\nu] + \left(\frac{k-2j-1}{4}\right)\mathsf{G}[\tfrac{k-1}{2}\,;\,\nu]\right\}\\ &\quad +\mathsf{G}[\tfrac{k+1}{2}\,;\,\nu] + 2(j-1)\,\mathsf{G}[j\,;\,\nu]\,.\end{aligned}$$

Since $\frac{d}{dx}\mathsf{G}[x;\nu]$, $0<\nu<\frac{1}{\sigma}$ is monotonically decreasing to zero as $x\to\infty$, it follows that $\phi(x)=\frac{G[x;\nu]}{x}$, $0<\nu<\frac{1}{\sigma}$ tends to zero as $x\to\infty$. Therefore, from the last two inequalities, it follows that

$$\frac{1}{k}\log|q_k|^{-1} \;\geq\; \frac{1}{2}\left(\mathsf{G}[k\,;\,\nu]\right) \qquad \text{as} \quad k\to\infty\,. \tag{2.3.4}$$

Using $\alpha(x)\in\Lambda$, $\beta(cx)\geq c\,\beta(x)$ for all $x>x_0(c)$ and (2.3.4), it follows that for all sufficiently large values of k,

$$\beta\left(\frac{1}{k}\log|q_k|^{-1}\right) \geq \frac{1}{2}\,\beta\left(\mathsf{G}[k\,;\,\nu]\right) \tag{2.3.5}$$

The above inequality, in view of (1.3.6), gives that $\rho(\alpha,\beta;B)\leq\frac{2}{\nu}$. □

Remark 2.3.3. *(i) We observe that our underlying assumption for this section,* $\sigma=\lim_{x\to\infty}\frac{\alpha(x)}{\beta(\log x)}>0$ *need not imply the condition* $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x}=\mathcal{O}(1)$, *as* $x\to\infty$, $0<c<\infty$ *in Theorem* 2.3.2. *For, consider* $\alpha(x)=\log x\in\Lambda$ *and* $\beta(x)=\sqrt{x}\in L^0$. *Then,* $\sigma>0$ *but* $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x}\to\infty$ *as* $x\to\infty$, *for all* $0<c<\infty$.

(ii)For a regualr C-fraction, if $|F_n|\leq\frac{\nu}{n^{\nu+1}}$, *it follows by a result of Thron [96] that the order (classical) of the entire function* $B(z)$ *(or* $A(z)$*) is atmost* $\frac{1}{\nu+1}$. *While, if* $\alpha(x)=\log x$, $\beta(x)=x$, *Theorem2.3.2 in this particular case gives that, for a general T-fraction, the order of the entire function* $B(z)$ *(or* $A(z)$*) is atmost* $\frac{2}{\nu}$. *However, our technique used for*

$B(z)$ being respectively the numerator and the denominator of the continued fraction $C(z)$. If $\alpha(x) \in \Lambda$, $\beta(x) \in \Lambda$ are such that

(i) $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x} = \mathcal{O}(1)$, $\quad (0 < c < \infty), \qquad$ *as* $x \to \infty$

(ii) $\mathsf{G}'[x\,;\,\nu] = \frac{d}{dx}(\mathsf{G}[x;\,\nu])$ *is monotonically decreasing for* $x > x_0$ *where* $\mathsf{G}[x\,;\,\nu] = \beta^{-1}(\nu\,\alpha\,(x))$ *and* $0 \le \frac{1}{\sigma} < \nu < \infty$ *and* σ *is defined by* (2.3.1)

(iii) $\beta(x\psi(x))/\beta(e^x) \to 0$ *as* $x \to \infty$, *for some function* $\psi(x)$ *tending to* ∞ *as* $x \to \infty$, *and, if, for* $n \ge j$, $|F_n| \le \dfrac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]}$ *and* $|G_n| \le \dfrac{\mathsf{G}'[n\,;\,\nu]}{\exp \mathsf{G}[n\,;\,\nu]}$, *then* $\rho(\alpha, \beta; f) \le \frac{1}{\nu}$.

Proof. We prove the theorem for $f \equiv B$. The proof for $f \equiv A$ is analogous.

Let $B(z) = \sum_{k=0}^{\infty} q_k\, z^k$.

Proceeding as in the proof of Theorem 2.3.2, and using the fact that $\alpha(x)$ and $\beta(x) \in \Lambda$, the inequality (2.3.4) gives that, for all sufficiently large values of k,

$$\beta\left(\frac{1}{k}\log|q_k|^{-1}\right) \ge \beta\left(\mathsf{G}[k\,;\,\nu]\right) .$$

Now, the above inequality, in view of (1.3.6), implies that

$$\rho\,(\alpha\,,\,\beta\,;B) \le \frac{1}{\nu}. \qquad \square$$

Remark 2.3.5. *Theorem 2.3.2 implies that if $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, then both the numerator and the denominator of a general T-fraction are of generalized $(\alpha\,,\,\beta)$- order atmost $(2/\nu)$, Theorem 2.3.3 under the stronger conditions $\alpha(x) \in \Lambda$ and $\beta(x) \in \Lambda$ gives an improved lower bound of generalized $(\alpha\,,\,\beta)$- order of $A(z)$ and $B(z)$ as $(1/\nu)$.*

Corollary 2.3.3. *Let* $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{F_n\, z}{1 + G_n\, z}\right)$, $F_n \neq 0$, *for* $n \ge 1$, *be a general T-fraction satisfying* (2.1.2). *Let* $E_n(A)$ *and* $E_n(B)$ *be defined by* (2.2.9) *and* $L_f^* = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(\frac{1}{n}\log\frac{1}{E_n(f)}\right)}$ *where* $f = A$ *or* B. *If* $\alpha(x) \in \Lambda$, $\beta(x) \in \Lambda$ *are such that*

(i) $\frac{d\,\beta^{-1}(c\,\alpha(x))}{d\,\log x} = \mathcal{O}(1) \quad (0 < c < \infty) \qquad$ *as* $x \to \infty$

(ii) $\mathsf{G}'[x\,;\,\nu] = \frac{d}{dx}(\mathsf{G}[x;\,\nu])$ *be monotonically decreasing for* $x > x_0$ *where* $\mathsf{G}[x\,;\,\nu] = \beta^{-1}(\nu\,\alpha\,(x))$, $0 \le \frac{1}{\sigma} < \nu < \infty$ *and* σ *is defined by* (2.3.1)

(iii) $\beta(x\psi(x))/\beta(e^x) \to 0$, *as* $x \to \infty$. *for some function* $\psi(x)$ *tending to* ∞ *as* $x \to \infty$, *and, if, for* $n \geq j$, $|F_n| \leq \frac{G'[n\,;\,\nu]}{\exp G[n\,;\,\nu]}$ *and* $|G_n| \leq \frac{G'[n\,;\,\nu]}{\exp G[n\,;\,\nu]}$, *then* $L_f^* \leq (1/\nu)$.

Proof. By Theorem 2.3.3, the numerator $A(z)$ and $B(z)$ of the continued fraction $C(z)$ are of generalized $(\alpha\,,\,\beta)$- order atmost $(1/\nu)$. Therefore, it follows by Theorem 1.3.1 that L_A^* and L_B^* are atmost $(1/\nu)$. □

Remark 2.3.6. *It follows that under the conditions of Corollary* 2.3.3, *the error in the polynomial approximation* $E_n(A)$ *(or* $E_n(B)$*) of the numerator* $A(z)$ *(or the denominator* $B(z)$*) of a general T-fraction is bounded above by*

$$E_n(A) \leq \frac{1}{\exp\left(n\,\beta^{-1}((\nu+\varepsilon)\;\alpha(n))\right)}$$

for $n \geq n_0 \equiv n_0(\varepsilon)$ *and* $\varepsilon > 0$.

2.4 Growth of the numerator and the denominator of a Modified general T-fraction

Let $f(z)$ be an entire function. It is well known that if $x \in$ R implies $f(x) \in$ R and $f(ix) \in$ R , then $f(z)$ is an even entire function (*i.e.*, $f(z) = f(-z)$ for all $z \in$ C). Some of the simple examples of even entire functions are $\cos z$ and $\sinh z/z$. In order to investigate the behaviour of an even function $f(z)$ on complex plane, it is sufficient to investigate the behaviour of $f(z)$ either in the left half plane or in the right half plane. An even entire functions of exponential type is reducible to the entire function of order 1/2. Therefore, for the even entire functions, the relationships among genus, maximum modulus and coefficients of its Taylor series are considerably simplified ([23], p35).

Let $f(z) = \sum\limits_{n=0 \text{ or } 1}^{\infty} a_n z^n$, $a_n \neq 0$ for all n, be an entire function. If $f(x) = f(-x)$ and $f(ix) = f(-ix)$ for all $x \in$ R then $a_1x + a_3x^3 + a_5x^5 + \cdots = 0$ and $a_1x - a_3x^3 + a_5x^5 - \cdots = 0$ for all $x \in$ R . This implies that $a_{2n+1} = 0$ for $n = 0, 1, 2, \ldots$. For a general T-fraction,

the coefficients of odd powers of z are non-zero in the Taylors series expansions of the limit functions $A(z)$ and $B(z)$. Therefore, it is not possible to obtain an even entire functions as a denominator (or numerator) of a separately convergent general T-fraction. In order to overcome this difficulty a new type of continued fraction is introduced.

Let

$$D(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z^2}{1 + G_n\, z^2} \right), \qquad F_n \neq 0 \tag{2.4.1}$$

be a *modified general T-fraction* satisfying

$$\sum_{n=1}^{\infty} |F_n| < \infty \qquad \sum_{n=1}^{\infty} |G_n| < \infty . \tag{2.4.2}$$

Let $A_n^*(z)$ and $B_n^*(z)$ denote the numerator and denominator of nth approximant of the continued fraction (2.4.1). In view of the recurrence relation $B_n^*(z) = (1 + G_n z^2) B_{n-1}^*(z) + F_n z^2 B_{n-2}^*(z)$, $n \geq 1$ with initial conditions $B_{-1} \equiv 0$ and $B_0 \equiv 1$, it follows that $B_n^*(z)$ (or similarly $A_n^*(z)$) is a polynomial of degree $2n$ and is an even function.

In the following theorem, for a modified general T-fraction, the convergence of the sequences $\{A_n^*(z)\}$ and $\{B_n^*(z)\}$ to an even entire functions are established.

Theorem 2.4.1. *Let* $D(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n z^2}{1 + G_n z^2} \right)$, $F_n \neq 0$ *for* $n \geq 1$ *be a modified general T-fraction satisfying (2.4.2). Then, the sequences* $\{A_n^*(z)\}_{n=1}^{\infty}$ *and* $\{B_n^*(z)\}_{n=1}^{\infty}$ *converge uniformly on compact subsets to even entire functions* $A^*(z)$ *and* $B^*(z)$ *respectively. Here the* $A_n^*(z)$ *and* $B_n^*(z)$ *are the numerator and the denominator of the nth approximant of the modified general T-fraction.*

Proof. Let $B_n^*(z) = \sum_{k=0}^{2n} q_k z^{2k}$. Then, $B_n^*(z)$ satisfies the three term recurrence relation

$$B_n^*(z) = (1 + G_n z^2) B_{n-1}^*(z) + F_n z^2 B_{n-2}^*(z), \qquad n \geq 1$$

with initial conditions $B_{-1}^* \equiv 0$ and $B_0^* \equiv 1$.

It is known [98] that if $\{W_n(z)\}$ satisfies

$$W_n(z) = (1 + d_n(z)) W_{n-1}(z) + g_n(z) W_{n-2}(z), \qquad n \geq 1, \qquad W_0 \equiv 1, \;\; W_{-1} \equiv 0$$

for $z \in \Delta$, and if $\sum_n |d_n(z)|$ and $\sum_n |g_n(z)|$ converge uniformly on compact subsets of Δ, then $\{W_n(z)\}$ converges uniformly on compact subsets of Δ to $W(z)$. Therefore, it follows that $\{B_n^*(z)\}$ converge uniformly on compact subsets of $\mathbb{C}$ to a entire function $B^*(z)$. Since each $B_n^*(z)$ is even function, the limit function is also an even function. The convergence of $\{A_n^*(z)\}$ to an even entire function $A^*(z)$ follows by analogous arguments. □

Theorem 2.4.1 gives that the condition (2.4.2) is sufficient for the separate convergence of the modified general T-fraction $D(z)$. In the following example, it is shown that the condition (2.4.2) is not necessary for the separate convergence of the modified general T-fraction.

Example 2.4.1. *Consider the modified general T-fraction*

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{-z^2/n}{1+(z^2/n)} \right) . \tag{2.4.3}$$

Let $A_n^*(z)$ and $B_n^*(z)$ denote the numerator and the denominator of the nth approximant of the continued fraction (2.4.3). It follows by the arguments analogous to those in Example 2.1.1, that the sequences $\{A_n^*(z)\}_{n=1}^{\infty}$ and $\{B_n^*(z)\}_{n=1}^{\infty}$ converge to even entire functions $A^*(z)$ and $B^*(z)$ respectively, $\sum |F_n| = \sum |G_n| = \sum \frac{1}{n} = \infty$.

For a separately convergent modified general T-fraction $D(z)$, the influence of elements F_n and G_n, $n = 1, 2, \cdots$, on the the order of $A^*(z)$ and $B^*(z)$ is described in the following:

Theorem 2.4.2. *Let $D(z) \equiv \underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{F_n\, z^2}{1+G_n\, z^2} \right)$, $F_n \neq 0$ for $n \geq 1$, be a modified general T-fraction satisfying (2.4.2). If, for $n \geq 1$, $|F_n| \leq r^n$ and $|G_n| \leq r^n$, $(0 < r < 1)$, then, the numerator $A^*(z)$ and the denominator $B^*(z)$ of the continued fraction $D(z)$ are even entire functions of order zero.*

Proof. We prove the theorem for $B^*(z) = \sum_{k=0}^{\infty} q_k z^{2k}$. The proof for $A^*(z)$ follows by analogous arguments.

Define, $C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right)$. By (2.4.2), the sequence $\{B_n(z)\}$ of denominators of the approximants of $C(z)$ converge uniformly to a entire function $B(z) = \sum_{k=0}^{\infty} a_k z^k$ (say). Further, $a_k = q_k$ for $k = 0, 1, 2, \cdots$. Using the condition $|F_n| \le \frac{1}{r^n}$ and $|G_n| \le \frac{1}{r^n}$, $n \ge 1$, and proceeding on the lines of proof of Theorem 2.1.1, the inequality (2.1.10) follows. Thus,

$$-\log|q_k| \ge -\log(k+1) + \frac{k^2}{4}\log\frac{1}{r} + \frac{k}{2}\log\frac{1}{r} + k\log(1-r) + a_k(r)$$

where, $a_k(r) = 0$ if k is even and $a_k(r) = (-1/4)\log r$ if k is odd. From the coefficient characterization of the order of entire function (c.f. (1.3.1) and the above inequality, it follows that the order ρ of $B^*(z)$ satisfies

$$\rho = \limsup_{k\to\infty} \frac{2k\log 2k}{-\log|q_k|} = 0 \qquad \square$$

The following Theorems 2.4.3 and 2.4.4 lead to sufficient conditions on the elements of a modified general T-fraction for its numerator and denominator to have generalized $(\alpha\,,\,\alpha)$- order equal to a prespecified constant $\mu \ge 1$.

Theorem 2.4.3. *Let* $D(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z^2}{1+G_n\, z^2}\right)$ *be a modified general T-fraction satisfying (2.4.2) and* $\rho(\alpha,\alpha;f)$ *and* $\lambda(\alpha,\alpha;f)$ *be the generalized* (α,α)*-order and the generalized lower* (α,α)*- order of* $f(z)$*, where* $f \equiv A^*$ *or* $f \equiv B^*$*;* $A^*(z)$ *and* $B^*(z)$ *being respectively the numerator and the denominator of the continued fraction* $D(z)$*. If* $\mathsf{G}'[x;\, \frac{1}{\mu}] \equiv \frac{d}{dx}\left(\mathsf{G}[x;\, \frac{1}{\mu}]\right)$ *monotonically decreases to zero as* $x \to \infty$*, where* $\mathsf{G}[x;\, \frac{1}{\mu}] = \alpha^{-1}(\frac{1}{\mu}\alpha(x))$*,* $\alpha(x) \in \Omega$*,* $1 \le \mu < \infty$*; and*

(i) if, for $n \ge 1$*,* $|F_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $|G_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$*, then* $\rho(\alpha,\alpha;f) \le \mu$*.*

(ii) if, for $n \ge 1$*,* $F_n \ge \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $G_n \ge \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$*, then* $\lambda(\alpha,\alpha;f) \ge \mu$*.*

Proof. We prove the theorem for $f \equiv B^*$; for $f \equiv A^*$ it follows by analogous arguments.

(i) Let $B^*(z) = \sum_{k=0}^{\infty} q_k\, z^{2k}$.

Using the condition $|F_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ and $|G_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, $n \ge 1$, and proceeding

on the same lines of proof as those of (i) of Theorem 2.2.1. we arrive at the inequalities (2.2.3) and (2.2.5). Thus, as $k \to \infty$,

$$\frac{1}{2k}\log|q_k|^{-1} \geq \begin{cases} O\left(\mathsf{G}[\frac{k}{2}-1\,;\,\frac{1}{\mu}]\right) & \text{if } k \text{ is even} \\ O\left(\mathsf{G}[\frac{k-1}{2}\,;\,\frac{1}{\mu}]\right) & \text{if } k \text{ is odd} \end{cases}$$

Since $\alpha(x) \in \Omega$, the above inequality implies that

$$\frac{1}{2k}\log|q_k|^{-1} \;\geq\; O\left(\mathsf{G}[k\,;\,\tfrac{1}{\mu}]\right) \qquad \text{as} \quad k \to \infty\,.$$

Using the property $\alpha(cx) \simeq \alpha(x)$, as $x \to \infty$ in the above inequality gives that

$$\tilde{L} \equiv \limsup_{k\to\infty} \frac{\alpha(2\,k)}{\alpha\left(\frac{1}{2k}\log|q_k|^{-1}\right)} \leq \limsup_{k\to\infty} \frac{\alpha(k)}{\alpha\left(A\;\mathsf{G}[k\,;\,\frac{1}{\mu}]\right)} \leq \mu\,.$$

and so, by (1.3.10)

$$\rho\,(\alpha\,,\,\alpha\,;B^*) \leq \mu\,.$$

(ii) If $F_n \;\geq\; \frac{1}{n^2\,\exp\,\mathsf{G}[n\,;\,\frac{1}{\mu}]}\,,\; n \geq 1$, then as in case (ii) of Theorem 2.2.1

$$q_k \;\geq\; \left(\frac{1}{2k}\right)^{2k} \frac{1}{\exp k\;\mathsf{G}[2k\,;\,\frac{1}{\mu}]}\,.$$

Thus,

$$\frac{1}{2k}\log|q_k|^{-1} \leq\; \log 2k + \frac{1}{2}\;\mathsf{G}[2k\,;\,\tfrac{1}{\mu}]\,.$$

Using the condition $\alpha(x) \in \Omega$, it follows in view of (1.3.13) and (1.3.14) that

$$\lambda(\alpha, \alpha; B^*) \geq \mu. \qquad \square$$

Theorem 2.4.4. *Let* $D(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}}\left(\frac{F_n\,z^2}{1+G_n\,z^2}\right)$ *be a modified general T-fraction satisfying (2.4.2) and* $\rho(\alpha, \alpha; f)$ *and* $\lambda(\alpha, \alpha; f)$ *be the generalized* (α, α)*-order and the generalized lower* (α, α)*- order of* $f(z)$*, where* $f \equiv A^*$ *or* $f \equiv B^*$*;* $A^*(z)$ *and* $B^*(z)$ *being respectively the numerator and the denominator of the continued fraction* $D(z)$*. If* $\mathsf{G}'[x;\,\frac{1}{\mu}] \equiv \frac{d}{dx}\left(\mathsf{G}[x;\,\frac{1}{\mu}]\right)$ *monotonically decreases to zero as* $x \to \infty$*, where* $\mathsf{G}[x;\,\frac{1}{\mu}] = \alpha^{-1}(\frac{1}{\mu}\alpha(x))$*,* $\alpha(x) \in \overline{\Omega}$*,* $1 \leq \mu < \infty$*; and*

(i) if, for $n \geq 1$, $|F_n| \leq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $|G_n| \leq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, *then* $\rho(\alpha, \alpha; f) \leq 1+\mu$.

(ii) if, for $n \geq 1$, $F_n \geq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $G_n \geq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, *then* $\lambda(\alpha, \alpha; f) \geq 1+\mu$.

Proof. We prove the theorem for $B^*(z)$; for $A^*(z)$ it follows by analogous arguments.

(i) Let $B^*(z) = \sum_{k=0}^{\infty} q_k\, z^{2k}$.

Using the condition $|F_n| \leq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ and $|G_n| \leq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, $n \geq 1$, and proceeding on the same lines of proof as those of (i) of Theorem 2.4.3, we get

$$\frac{1}{2k} \log |q_k|^{-1} \;\geq\; O\left(\mathsf{G}[k;\, \tfrac{1}{\mu}]\right) \qquad \text{as} \quad k \to \infty\, .$$

Using the above inequality and the condition $\alpha(x) \in \overline{\Omega}$, it follows that

$$\rho\,(\alpha\,,\, \alpha\, ; B^*) \leq 1 + \mu\, .$$

(ii) If $F_n \geq \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, $n \geq 1$, then as in case (ii) of Theorem 2.4.3

$$q_k \;\geq\; \left(\frac{1}{2k}\right)^{2k} \frac{1}{\exp k\;\; \mathsf{G}[2k;\, \frac{1}{\mu}]}\, .$$

Thus,

$$\frac{1}{2k} \log |q_k|^{-1} \leq \log 2k + \frac{1}{2}\; \mathsf{G}[2k;\, \tfrac{1}{\mu}]\, .$$

Using the condition $\alpha(x) \in \overline{\Omega}$, it follows in view of (1.3.13) and (1.3.14) that

$$\lambda(\alpha, \alpha; B^*) \geq 1 + \mu. \qquad \square$$

Let $A^*(x)$ and $B^*(x)$ be the restrictions on the real line of the numerator $A^*(z)$ and the denominator $B^*(z)$ of a modified general T-fraction satisfying (2.4.2). Define, for $n = 0, 1, 2, \cdots$,

$$E_n(A^*) = \inf_{Q \in \pi_n} ||A^* - Q|| \qquad \text{and} \qquad E_n(B^*) = \inf_{Q \in \pi_n} ||B^* - Q|| \tag{2.4.4}$$

where, for $f \equiv A^*$ or B^*, $||f - Q|| = \sup_{-1 \leq x \leq 1} |f(x) - Q(x)|$ and π_n denotes the set of polynomials of degree atmost n. The following theorem leads to an upper bound on the error of approximations $E_n(A^*)$ and $E_n(B^*)$.

Theorem 2.4.5. *Let* $D(z) \equiv \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\dfrac{F_n z^2}{1 + G_n z^2} \right)$ *be a modified general T-fraction satisfying (2.4.2). Let* $E_n(A)$ *and* $E_n(B)$ *be defined by (2.4.4) and* $L_f = \limsup \frac{\alpha(n)}{\alpha\left(\frac{1}{n}\log \frac{1}{E_n(f)}\right)}$, *where* $f \equiv A^*$ *or* $f \equiv B^*$; $A^*(z)$ *and* $B^*(z)$ *being respectively the numerator and the denominator of the continued fraction* $D(z)$. *If* $\mathsf{G}'[x;\, \frac{1}{\mu}] \equiv \frac{d}{dx}\left(\mathsf{G}[x;\, \frac{1}{\mu}]\right)$ *monotonically decreases to zero as* $x \to \infty$, *where* $\mathsf{G}[x;\, \frac{1}{\mu}] = \alpha^{-1}(\frac{1}{\mu}\alpha(x))$; $\alpha(x) \in \Omega$ *or* $\overline{\Omega}$, $1 \le \mu < \infty$; *and*

(i) if, for $n \ge 1$, $|F_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $|G_n| \le \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, *then* $P(L_f) \le \mu$ *for* $\alpha(x) \in \Omega$ *and* $P(L_f) \le 1 + \mu$ *for* $\alpha(x) \in \overline{\Omega}$.

(ii) if, for $n \ge 1$, $F_n \ge \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$ *and* $G_n \ge \frac{1}{n^2 \exp \mathsf{G}[n;\, \frac{1}{\mu}]}$, *then* $P(L_f) \ge \mu$ *for* $\alpha(x) \in \Omega$ *and* $P(L_f) \ge 1 + \mu$ *for* $\alpha(x) \in \overline{\Omega}$.

Here, $P(\Psi) = \max\{1, \Psi\}$ *if* $\alpha(x) \in \Omega$, $P(\Psi) = 1 + \Psi$ *if* $\alpha(x) \in \overline{\Omega}$.

Proof. In view of Theorem 2.4.3(i), Theorem 2.4.4(i) and Theorem 1.3.2, it follows that $P(L_A)$ and $P(L_B)$ are atmost μ if $\alpha(x) \in \Omega$ and atmost $1 + \mu$ if $\alpha(x) \in \overline{\Omega}$. Similarly, by Theorem 2.4.3(ii), Theorem 2.4.4(ii) and Theorem 1.3.2, (ii) follows. □

Remark 2.4.1. *Under the conditions (i) and (ii), it is easily seen by Theorem 2.4.5 that the error in the polynomial approximation* $E_n(A^*)$ *(or* $E_n(B^*)$*) of the numerator* $A^*(z)$ *(or the denominator* $B^*(z)$*) of a general T-fraction is bounded above by*

$$E_n(A^*) \;\le\; \exp\left(-n\,\alpha^{-1}\left(\frac{1}{\mu+\varepsilon}\alpha(n)\right)\right) \qquad \text{if } \alpha(x) \in \Omega \text{ or } \overline{\Omega}$$

for $n \ge n_0 \equiv n_0(\varepsilon)$ *and* $\varepsilon > 0$.

Chapter 3

Dynamics of entire functions of slow growth arising from separately convergent continued fractions

In the present chapter, we investigate the dynamics of the entire transcendental functions of slow growth arising as the numerator and the denominator of a separately convergent general T-fraction. Let $A(z)$ and $B(z)$ be the numerator and the denominator (c.f. Section 1.2) of a separately convergent general T-fraction. We introduce one parameter families $\mathcal{A} \equiv \{A_\lambda(z) = \lambda A(z) : \lambda > 0\}$ and $\mathcal{B} \equiv \{B_\lambda(z) = \lambda B(z) : \lambda > 0\}$, and study the dynamics of the functions belonging to these families. The results obtained herein are used to computationally generate the pictures of Julia sets of the functions $A_\lambda(z)$ and $B_\lambda(z)$.

3.1 One parameter families $\mathcal{A}$ and $\mathcal{B}$

Let

$$C(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n\, z}{1+G_n\, z}\right), \qquad F_n \neq 0 \tag{3.1.1}$$

be a general T-fraction satisfying the condition

$$0 < F_n \leq \frac{1}{n^2 \exp n^{\frac{1}{\mu}}}, \qquad 0 < G_n \leq \frac{1}{n^2 \exp n^{\frac{1}{\mu}}}, \quad n \geq 1 \tag{3.1.2}$$

where, $1 < \mu < 2$.

By three term recurrence relations (1.2.2), it is easily seen that

$$A_n(z) = F_n\ z\ A_{n-2}(z) + (1 + G_n\ z)A_{n-1}(z)$$

$$B_n(z) = F_n\ z\ B_{n-2}(z) + (1 + G_n\ z)B_{n-1}(z)$$

with initial conditions $A_{-1} \equiv 1$, $A_0 \equiv 0$, $B_{-1} \equiv 0$ and $B_0 \equiv 1$, where $A_n(z)$ and $B_n(z)$ denote the numerator and the denominator respectively of nth approximant of the general T-fraction (3.1.1). Thus, $A_n(z) = \sum_{k=1}^{n} p_k^{(n)} z^k$ is a polynomial of degree n with $p_1^{(n)} = F_1$ for $n \geq 1$ and $p_k^{(n)} > 0$ for all $n \geq 1$ and $k \geq 1$. Similarly, $B_n(z) = \sum_{k=0}^{n} q_k^{(n)} z^k$ is a polynomial of degree n with $q_0^{(n)} = 1$ for $n \geq 1$ and $q_k^{(n)} > 0$ for all $n \geq 1$ and $k \geq 0$. By (3.1.2) and Theorem 1.2.2, it follows that the sequences $\{A_n(z)\}$ and $\{B_n(z)\}$ converge uniformly on every compact subset of $\mathbb{C}$ to the entire functions $A(z)$ and $B(z)$. Therefore, $A(z)$ and $B(z)$ can be written as

$$A(z) = \sum_{k=1}^{\infty} p_k z^k, \qquad p_1 = F_1 \text{ and } 0 < p_k = \lim_{n\to\infty} p_k^{(n)} \text{ for all } k \geq 1 \tag{3.1.3}$$

$$B(z) = \sum_{k=0}^{\infty} q_k z^k, \qquad q_0 = 1 \text{ and } 0 < q_k = \lim_{n\to\infty} q_k^{(n)} \text{ for all } k \geq 0 \tag{3.1.4}$$

By Theorem 2.2.2 and (3.1.2), it is easily seen that the entire functions $A(z)$ and $B(z)$ are of generalized (log, log)- order atmost $1+\mu$, $1 < \mu < 2$. Consequently, the order (classical) of $A(z)$ and $B(z)$ is zero *i.e.* these functions are of slow growth.

Let

$$\mathcal{A} \equiv \{A_\lambda(z) = \lambda A(z) : \lambda > 0 \text{ and } A(z) \text{ is given by } (3.1.3)\}$$

and

$$\mathcal{B} \equiv \{B_\lambda(z) = \lambda B(z) : \lambda > 0 \text{ and } B(z) \text{ is given by } (3.1.4)\}$$

be one parameter families of entire transcendental functions. The present chapter is devoted to the study of the dynamics of the functions $A_\lambda \in \mathcal{A}$ and $B_\lambda \in \mathcal{B}$. In Section 3.2,

the dynamics of $A_\lambda \in \mathcal{A}$ is described. Firstly, the existence and nature of the fixed points of $A_\lambda(z)$ in the positive real line is investigated and the dynamics of $A_\lambda(x)$ for $x \geq 0$ is studied. Let $\lambda_A^* = \frac{1}{A'(0)}$. If $0 < \lambda < \lambda_A^*$, it is found in this section that the Fatou set of $A_\lambda(z)$ contains the basin of attraction (c.f. Theorem 1.1.7) of the real attracting fixed point 0 and a general description of the basin of attraction A(0) is given. Further, in this section, the dynamics of $A_\lambda(z)$ for $z \in \mathbb{C}$ is described for the three different cases, viz, $0 < \lambda < \lambda_A^*$, $\lambda = \lambda_A^*$ and $\lambda > \lambda_A^*$. In all the three cases, we obtain computationally useful characterization of the Julia set of $A_\lambda(z)$ as the closure of the set of points with orbits escaping to infinity under iteration of A_λ. The dynamics of $B_\lambda \in \mathcal{B}$ is studied in Section 3.3, wherein, the nature of the fixed points and the dynamics of $B_\lambda(z)$ on the positive real line are investigated. Next, a description of the basin of attraction of the real attracting fixed point a_λ of the entire function $B_\lambda(z)$ is found for $0 < \lambda < \lambda_B^* = \frac{1}{B'(x^*)}$; x^* being the unique positive real root of the equation $B(x) - xB'(x) = 0$. Similarly, a description of the parabolic domain (c.f. Theorem 1.1.7) corresponding to the rationally indifferent fixed point x^* of $B_\lambda(z)$ is found for $\lambda = \lambda_B^*$. Further, in this section, the dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ is described for all the three different cases, viz, $0 < \lambda < \lambda_B^*$, $\lambda = \lambda_B^*$ and $\lambda > \lambda_B^*$ and, analogous to that of $A_\lambda(z)$, a computationally useful characterization of the Julia set of $B_\lambda(z)$ is obtained. In Section 3.4, the characterizations of the Julia set found in Sections 3.2 and 3.3 is applied to computationally generate the pictures of the Julia sets of $A_\lambda(z) = \lambda A(z)$ and $B_\lambda(z) = \lambda B(z)$ for different values of λ, where $A(z)$ and $B(z)$ are the numerator and the denominator respectively of the separately convergent general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{z/\left(n^2 \exp\left(n^{\frac{1}{\mu}}\right)\right)}{1 + z/\left(n^2 \exp\left(n^{\frac{1}{\mu}}\right)\right)} \right)$ with $\mu = 1.5$.

3.2 Dynamics of slow growth entire function $A_\lambda \in \mathcal{A}$

We begin by studying the dynamics of $A_\lambda(z)$ on the positive real line. The nature of the fixed points of $A_\lambda(x)$ for $x \geq 0$ is given in Theorem 3.2.1 and the dynamics of $A_\lambda(x)$ on the positive real line is described in Theorem 3.2.2.

Throughout in this section, we denote

$$\lambda_A^* = \frac{1}{A'(0)} = \frac{1}{F_1} \tag{3.2.1}$$

Theorem 3.2.1. *Let $A_\lambda(x) = \lambda A(x)$ for $x \geq 0$, where $A(z)$ is defined by (3.1.3).*

(a) *If $0 < \lambda < \lambda_A^*$, $A_\lambda(x)$ has an attracting fixed point at $x = 0$ and a repelling fixed point at $x = r_\lambda > 0$.*

(b) *If $\lambda = \lambda_A^*$, $A_\lambda(x)$ has a unique rationally indifferent fixed point at $x = 0$.*

(c) *If $\lambda > \lambda_A^*$, $A_\lambda(x)$ has a repelling fixed point at $x = 0$.*

Proof. Define $g_\lambda(x) = A_\lambda(x) - x = \lambda A(x) - x$ for $x \in \mathrm{R}$. The zeros of $g_\lambda(x)$ are fixed points of $A_\lambda(x)$. It is easily seen that $g_\lambda(x)$ is continuously differentiable in R and positive for x sufficiently large. The function $g_\lambda'(x)$ is strictly increasing for $x \geq 0$ and $g_\lambda'(x) \to \infty$ as $x \to \infty$.

(a) Since $\lambda_A^* = 1/A'(0)$ and $A'(x)$ is strictly increasing function for $x \geq 0$, $g_\lambda'(x)$ is a strictly increasing function for $x \geq 0$ and $g_\lambda'(0) < 0$ for $0 < \lambda < \lambda_A^*$. Using the continuity of $g_\lambda'(x)$, it follows that, for $0 < \lambda < \lambda_A^*$, there exist a unique real number $\tilde{x} \equiv \tilde{x}(\lambda)$ such that $g_\lambda'(\tilde{x}) = 0$, $g_\lambda'(x) < 0$ for $0 \leq x < \tilde{x}$ and $g_\lambda'(x) > 0$ for $x > \tilde{x}$. Therefore, in view of $g_\lambda''(\tilde{x}) > 0$, $g_\lambda(x)$ attains a unique local minimum value in $[0, \infty)$ at $x = \tilde{x}$. The function $\phi(x) = A(x) - xA'(x)$, is strictly decreasing in the interval $[0, \infty)$ and $\phi(0) = 0$. This implies that $\phi(x) < 0$ for $x > 0$. Since $\tilde{x} > 0$, $0 > \phi(\tilde{x}) = A(\tilde{x}) - \tilde{x}A'(\tilde{x})$ $= A'(\tilde{x})\, g_\lambda(\tilde{x})$. Consequently, $g_\lambda(\tilde{x}) < 0$ since $A'(\tilde{x}) > 0$. Now, as $g_\lambda(x)$ attains local minimum at $\tilde{x}$ and $g_\lambda(\tilde{x}) < 0$, the function $g_\lambda(x)$ has only two zeros a_λ and r_λ (say)

with $a_\lambda < \tilde{x} < r_\lambda$. Further, $g_\lambda(0) = 0$ implies that $a_\lambda = 0$. Since $0 = a_\lambda < \tilde{x} < r_\lambda$ $g'_\lambda(\tilde{x}) = 0$, $g'_\lambda(a_\lambda) < 0$ and $g'_\lambda(r_\lambda) > 0$, it follows that $A'_\lambda(a_\lambda) < 1$ and $A'_\lambda(r_\lambda) > 1$. Thus, the point $a_\lambda = 0$ is an attracting fixed point and the point r_λ is a repelling fixed point of $A_\lambda(x)$. This proves (a).

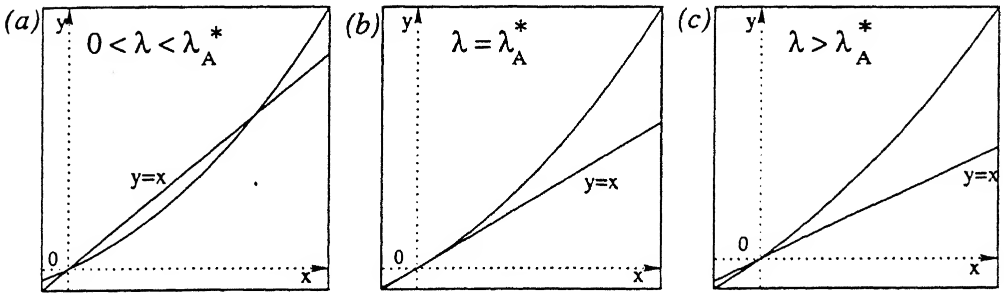


Figure 3.1: The graphs of $A_\lambda(x)$ for (a) $0 < \lambda < \lambda_A^*$, (b) $\lambda = \lambda_A^*$ and (c) $\lambda > \lambda_A^*$.

(b) If $\lambda = \lambda_A^*$, $g_\lambda(0) = 0$ and $g'_\lambda(0) = 0$ so that $A'_\lambda(0) = 1$. Thus, $A_\lambda(x)$ has a unique rationally indifferent fixed point at $x = 0$. Since $g'_{\lambda_A^*}(x)$ is strictly increasing for $x \geq 0$, it follows that $g_{\lambda_A^*}(x) > 0$ for $x > 0$. Consequently, $A_{\lambda_A^*}(x)$ has no fixed points for $x > 0$. This proves (b).

(c) If $\lambda > \lambda_A^*$ then $g_\lambda(0) = 0$ and $g'_\lambda(0) = \lambda A'(0) - 1 > 0$. Therefore, $x = 0$ is a repelling fixed point for $A_\lambda(x)$. Since $g'_\lambda(x)$ is positive for $x \geq 0$ and $g_\lambda(0) = 0$, it follows that $g_\lambda(x) > 0$ for $x > 0$. Thus, $A_\lambda(x)$, for $x \geq 0$ has only one repelling fixed point at $x = 0$, completing the proof of (c). $\square$

In the real dynamics, a geometric picture of the behavior of all orbits of a system is provided by the *phase portrait*. It consists of a diagram representing possible begining positions in the system and arrows which indicate the change in these positions under iteration of a function.

The following theorem describes the dynamics of $A_\lambda(x)$ for $x \geq 0$ and also provides a phase portrait of $A_\lambda(x)$:

Theorem 3.2.2. *Let $A_\lambda(x) = \lambda A(x)$ for $x \in$ R, where $A(z)$ is given by (3.1.3).*

Theorem 3.2.2. *Let $A_\lambda(x) = \lambda A(x)$ for $x \in$ R , where $A(z)$ is given by (3.1.3).*

(a) *If $0 < \lambda < \lambda_A^*$, there exists a negative real number $x_0 \equiv x_0(\lambda)$ such that $A_\lambda^n(x) \to 0$ as $n \to \infty$ for $x_0 < x < r_\lambda$ and $A_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$, where 0 and r_λ are the attracting and the repelling fixed points of $A_\lambda(x)$ respectively.*

(b) *If $\lambda = \lambda_A^*$, there exists a negative real number $x_0 \equiv x_0(\lambda)$ such that $A_\lambda^n(x) \to 0$ as $n \to \infty$ for $x_0 < x \leq 0$ and $A_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x > 0$, where 0 is the rationally indifferent fixed point of $A_\lambda(x)$.*

(c) *If $\lambda > \lambda_A^*$, $A_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x > 0$.*

Proof. (a) If $0 < \lambda < \lambda_A^*$, by Theorem 3.2.1(a), it follows that $A_\lambda(x)$ has an attracting fixed point $a_\lambda = 0$ and a repelling fixed point r_λ (say) with $a_\lambda < r_\lambda$. Since $0 < A_\lambda'(0) < 1$ for $0 < \lambda < \lambda_A^*$, $A_\lambda'(x)$ is strictly increasing for $x \geq 0$ and, due to continuity of $A_\lambda'(x)$ for $x \in$ R , there exists a negative real number $x_0 \equiv x_0(\lambda)$ such that $0 < A_\lambda'(x) < 1$ for $x_0 < x < 0$.

Now, as in the proof of Theorem 3.2.1(a), $g_\lambda(x) = A_\lambda(x) - x$ has only two zeros at $a_\lambda = 0$ and r_λ, and $g_\lambda(\tilde{x}) < 0$ so that

$$g_\lambda(x) = A_\lambda(x) - x \begin{cases} > 0 & \text{for } x \in (r_\lambda, \infty) \\ < 0 & \text{for } x \in (0, r_\lambda) \end{cases} \tag{3.2.2}$$

Since, in view of (3.1.3), $A_\lambda(x) > 0$ for $x > 0$ and by (3.2.2), it follows that, for $0 < x < r_\lambda$,

$$|A_\lambda(x)| < |x| . \tag{3.2.3}$$

If $x_0 < x < 0$ then, by the mean value theorem, $|A_\lambda(x)| \leq A_\lambda'(c)|x|$, where $x_0 < x < c < 0$. Since $0 < A_\lambda'(x) < 1$ for $x_0 < x < 0$, it follows that $A_\lambda'(c) < 1$. Consequently, $|A_\lambda(x)| < |x|$ for $x_0 < x < 0$. This inequality together with the inequality (3.2.3) gives that, for $x_0 < x < r_\lambda$ and $x \neq 0$, $|A_\lambda(x)| < |x|$. Thus, for $x_0 < x < r_\lambda$, $A_\lambda^n(x) \to 0$ as $n \to \infty$. Further, if $x > r_\lambda$, $A_\lambda^n(x) \to \infty$ as $n \to \infty$, since $A_\lambda(x) > x$ for $x > r_\lambda$. This completes the proof of (a).

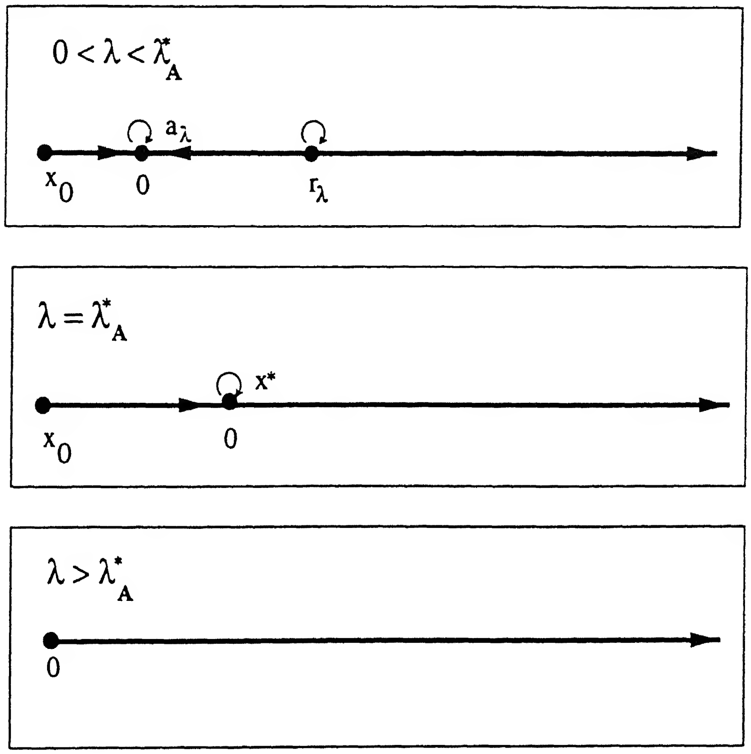


Figure 3.2: Phase portrait of the function $A_\lambda(x) = \lambda A(x)$ for $x \geq 0$ and $\lambda > 0$.

(b) By Theorem 3.2.1(b), if $\lambda = \lambda^*_A$, $A_\lambda(x)$ has a unique rationally indifferent fixed point at $x = 0$. Since $A'_\lambda(x)$ is strictly increasing for $x \geq 0$, $A'_{\lambda^*_A}(0) = 1$, due to continuity of $A'_\lambda(x)$ for $x \in$ R , there exists a negative real number $x_0 \equiv x_0(\lambda)$ such that $0 < A'_{\lambda^*_A}(x) < 1$ for $x_0 < x < 0$. Since $A'_{\lambda^*_A}(x) < 1$ for $x_0 < x < 0$ and $A'_{\lambda^*_A}(0) = 1$, it follows that if $x_0 < x < 0$, $|A_{\lambda^*_A}(x)| < |x|$. Therefore, $A^n_{\lambda^*_A}(x) \to 0$ as $n \to \infty$ for $x_0 < x < 0$. If $x > 0$ then $A_{\lambda^*_A}(x) > x$ and hence $A^n_{\lambda^*_A}(x) \to \infty$ as $n \to \infty$ for $x > 0$. This proves (b).

(c) If $\lambda > \lambda^*_A$, for $x > 0$, $A^n_\lambda(x) \to \infty$ as $n \to \infty$, since $A_\lambda(x) > x$ for $\lambda > \lambda^*_A$, completing the proof of (c). □

3.2.I Dynamics of $A_\lambda(z)$ for $z \in$ C and $0 < \lambda < \lambda^*_A$

In this subsection, the dynamics $A_\lambda(z)$ for $z \in$ C and $0 < \lambda < \lambda^*_A$ where λ^*_A is defined by (3.2.1), is described.

Let A(0) be the basin of attraction of the attracting fixed point 0 of $A_\lambda(z)$ *i.e.*,

$$\mathsf{A}(0) = \{z \in \mathbb{C} : A_\lambda^n(z) \to 0 \text{ as } n \to \infty\} .$$

We find in Proposition 3.2.1 that the basin of attraction A(0) contains all the points which get mapped by $A_\lambda(z)$ inside the open disk centered at 0 and having radius $\tilde{x}$, where $\tilde{x}$ is the unique positive real root of the equation $A'_\lambda(x) = 1$. For $0 < \lambda < \lambda_A^*$, Theorem 3.2.3 gives computationally useful characterization of the Julia set $\mathcal{J}(A_\lambda)$ as the closure of the set of escaping points of $A_\lambda(z)$.

Proposition 3.2.1. *Let $A_\lambda \in \mathcal{A}$ and $0 < \lambda < \lambda_A^*$. Then, the basin of attraction* A(0) *of the real attracting fixed point* 0 *of $A_\lambda(z)$ contains the set $D = \{z \in \mathbb{C} : |A_\lambda(z)| < \tilde{x}\}$, where $\tilde{x} > 0$ is the unique real number such that $A'_\lambda(\tilde{x}) = 1$.*

Proof. Let $A_\lambda(z) = \lambda A(z)$ for $z \in \mathbb{C}$. Since, by (3.1.3), $A'(x)$ is strictly increasing for $x > 0$ and tends to ∞, and $A'_\lambda(0) = \lambda A'(0) = \frac{\lambda}{\lambda_A^*} < 1$ for $0 < \lambda < \lambda_A^*$, it follows that there exists a unique $\tilde{x} > 0$ such that $A'_\lambda(\tilde{x}) = 1$.

We first show that $A_\lambda(z)$ maps the open disk $D_{\tilde{x}}(0)$, centered at origin and having radius $\tilde{x}$ into itself. Define $\phi(x) = A(x) - xA'(x)$. As in the arguments for the proof of Theorem 3.2.1(a), $\phi(0) = 0$ and $\phi(x) < 0$ for $x > 0$. Since $A'(\tilde{x})\left(\frac{A(\tilde{x})}{A'(\tilde{x})} - \tilde{x}\right) = A(\tilde{x}) - \tilde{x}A'(\tilde{x}) = \phi(\tilde{x}) < 0$, it follows that $\frac{A(\tilde{x})}{A'(\tilde{x})} < \tilde{x}$. Consequently, $A_\lambda(x) = \lambda A(x) = \frac{1}{A'(\tilde{x})}A(x) < \frac{A(\tilde{x})}{A'(\tilde{x})} < \tilde{x}$, for $x < \tilde{x}$. Therefore, $\max_{|z|=\tilde{x}} |A_\lambda(z)| \leq \max_{|z|=\tilde{x}} A_\lambda(|z|) = A_\lambda(\tilde{x}) \leq \frac{A(\tilde{x})}{A'(\tilde{x})} < \tilde{x}$. Now, the maximum modulus principle gives $|A_\lambda(z)| < \tilde{x}$ for $z \in D_{\tilde{x}}(0)$. Thus, $A_\lambda(z)$ maps the open disk $D_{\tilde{x}}(0)$ into itself.

Since $D_{\tilde{x}}(0)$ is a simply connected domain and $A_\lambda(D_{\tilde{x}}(0)) \subseteq D_{\tilde{x}}(0)$, by Schwarz lemma ([30], p264), $A_\lambda^n(z) \to 0$ as $n \to \infty$ for all $z \in D_{\tilde{x}}(0)$. It is easily seen that $A_\lambda(z)$ maps D into $D_{\tilde{x}}(0)$ and therefore, $A_\lambda^n(z) \to 0$ as $n \to \infty$ for all $z \in D$. Thus, $\mathsf{A}(0) \supset D$. □

Remark 3.2.1. *(i) By Theorem* 3.2.2*(a), for* $0 < \lambda < \lambda_A^*$, *there exists a negative real number $x_0 \equiv x_0(\lambda)$ such that $A_\lambda^n(x) \to 0$ as $n \to \infty$ for $x_0 < x < r_\lambda$, where $r_\lambda > 0$ is a*

repelling fixed point of $A_\lambda(z)$. *Therefore, it follows that the interval* $(x_0,\ r_\lambda\ )$ *is contained in the basin of attraction* $\mathsf{A}(0)$ *of the attracting fixed point* $x = 0$ *of* $A_\lambda(z)$, $0 < \lambda < \lambda_A^*$.

(ii) By the arguments in the proof of Proposition 3.2.1, *it follows that* $A_\lambda(z)$ *maps the open disk* $D_{\tilde{x}}(0)$, *centered at origin and having radius* $\tilde{x}$ *into itself. Therefore,*

$$D_{\tilde{x}}(0) \subseteq D \subseteq \mathsf{A}(0) \subseteq \mathcal{F}(A_\lambda)\ .$$

For $0 < \lambda < \lambda_A^*$, by Theorem 3.2.1(a), r_λ is the repelling fixed point for $A_\lambda(z)$ and therefore r_λ belongs to the Julia set $\mathcal{J}(A_\lambda)$. It is well known that the Julia set of an entire function is characterized as the closure of the set of repelling periodic point (c.f. Theorem 1.1.5). In the following theorem, we find that the Julia set $\mathcal{J}(A_\lambda)$ is the closure of the set of escaping points.

Theorem 3.2.3. *Let* $A_\lambda \in \mathcal{A}$ *and* $Esc(A_\lambda) = clo\ \{z \in \mathbb{C} : A_\lambda^n(z) \to \infty \text{ as } n \to \infty\ \}$ *be the closure of the set of escaping points of* $A_\lambda(z)$. *If* $0 < \lambda < \lambda_A^*$, *then the Julia set* $\mathcal{J}(A_\lambda) = Esc(A_\lambda)$.

Proof. Let $z_0 \in \mathcal{J}(A_\lambda)$ and U be any neighborhood of z_0. Since $\{A_\lambda^n\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \{A_\lambda^n(U)\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > r_\lambda$ such that $\hat{x} \in \bigcup_n \{A_\lambda^n(U)\}$, where r_λ is the repelling fixed point of $A_\lambda(z)$. Therefore, there exists a point $\hat{z} \in U$ such that $A_\lambda^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 3.2.2(a), it follows that $A_\lambda^n(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > r_\lambda$. Thus, there exists a point $\hat{z} \in U$ such that $A_\lambda^n(\hat{z}) \to \infty$ as $n \to \infty$ and $z_0 \in Esc(A_\lambda)$ follows. Consequently, $\mathcal{J}(A_\lambda) \subseteq Esc(A_\lambda)$.

Let $z_1 \in Esc(A_\lambda)$. Without loss of generality we assume that $A_\lambda^n(z_1) \to \infty$ as $n \to \infty$. Suppose $z_1 \notin \mathcal{J}(A_\lambda)$. Then, there exists a neighborhood V of z_1 such that $\{A_\lambda^n\}$ is normal in V *i.e.* $V \subset \mathcal{F}(A_\lambda)$. This means that $\{A_\lambda^n\}$ contains either (i) a subsequence which converges to a limit function $g \not\equiv \infty$ uniformly on each compact subset of V, or (ii) a subsequence which converges uniformly to ∞ on each compact subset of V.

CASE (I): Let $A_\lambda^{n_k}(z) \to g(z)$ for all $z \in V$ as $n_k \to \infty$. By Weirstrass theorem, the function $g(z)$ is analytic in V. Let w be any point in V. Then, $w \in \mathcal{F}(A_\lambda)$ and so it must lie in some component of $\mathcal{F}(A_\lambda)$. Since the generalized (log, log)- order of $A_\lambda(z)$ is $1+\mu$, where $1<\mu<2$, it follows that by Theorem 1.3.8, every component of the Fatou set $\mathcal{F}(A_\lambda)$ is bounded. It therefore follows that $|A_\lambda^n(w)| < M$, $n = 1, 2, \cdots$, for some M and hence $|g(w)| < M$. But, the forward orbit of the point $z_1 \in V$ escapes to ∞ under iteration of A_λ. Thus, as $n_k \to \infty$, $A_\lambda^{n_k}(z_1) \to \infty$, while $A_\lambda^{n_k}(w) \to g(w)$ with $|g(w)| < M$ for $w \in V$. This implies that $\{A_\lambda^n\}$ is not normal at z_1.

CASE (II): If $A_\lambda^{n_k}(z) \to \infty$ for all $z \in V \subset \mathcal{F}(A_\lambda)$ as $n_k \to \infty$, then for each M^* sufficiently large we can find a point $z^* \in V$ such that $|A_\lambda^N(z^*)| > M^*$ for some positive integer N. By invariance property of the Fatou set, the point $A_\lambda^N(z^*)$ must lie in some component of the Fatou set of $A_\lambda(z)$. It therefore follows that atleast one component of the Fatou set $\mathcal{F}(A_\lambda)$ is unbounded. This contradicts the fact that every component of the Fatou set $\mathcal{F}(A_\lambda)$ is bounded (c.f. Theorem 1.3.8). Therefore, $\{A_\lambda^n\}$ is not normal at z_1 and so $z_1 \in \mathcal{J}(A_\lambda)$.

By case (I) and case (II) it follows that $Esc(A_\lambda) \subseteq \mathcal{J}(A_\lambda)$. □

Remark 3.2.2. *(i) Theorem* 3.2.3 *provides a characterization of the Julia set of $A_\lambda(z)$ as the closure of the set of all escaping points of $A_\lambda(z)$. Such a characterization, hitherto known only for certain critically finite entire transcendental functions [37], is quite useful in computationally generating the pictures of the Julia set of $A_\lambda(z)$ in Section 3.4.*

(ii) By Theorem 3.2.2*(a) and Theorem* 3.2.3*, it follows that the interval (r_λ, ∞) is contained in the Julia set $\mathcal{J}(A_\lambda)$ for $0<\lambda<\lambda_A^*$, where $r_\lambda > 0$ is the repelling fixed point of $A_\lambda(z)$.*

3.2.II Dynamics of $A_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda = \lambda_A^*$

The following proposition shows that if $\lambda = \lambda_A^*$, where λ_A^* is defined by (3.2.1), the Fatou set of $A_\lambda(z)$ contains a parabolic domain.

Proposition 3.2.2. *Let $A_\lambda \in \mathcal{A}$ and $\lambda = \lambda_A^*$. Then, the Fatou set $\mathcal{F}(A_\lambda)$ contains a parabolic domain.*

Proof. Let $\mathsf{U} = \left\{z \in \mathbb{C} : A_{\lambda_A^*}^n(z) \to 0 \text{ as } n \to \infty\right\}$. By Theorem 3.2.1(b), it follows that $A_\lambda(z)$ has a rationally indifferent fixed point at $x = 0$. Since by Theorem 3.2.2(b), there exist $x_0 \equiv x_0(\lambda_A^*)$ such that $A_{\lambda_A^*}^n(x) \to 0$ as $n \to \infty$ for $x_0 < x < 0$ and the positive real points tend to ∞ under iteration of $A_{\lambda_A^*}$, the rationally indifferent fixed point 0 lies on the boundary of U. Thus, U is a parabolic domain in the Fatou set of $A_{\lambda_A^*}(z)$. □

Remark 3.2.3. *As in the proof of Proposition 3.2.2, $A_\lambda^n(x) \to 0$ for $x_0 < x < 0$ and $A_\lambda^n(x) \to \infty$ for $x > 0$ as $n \to \infty$, when $\lambda = \lambda_A^*$. Therefore, the indifferent fixed point $x = 0$ belongs to the Julia set of $A_{\lambda_A^*}(z)$ and the interval $(x_0, 0)$ is contained in the parabolic domain U. Further, the rationally indifferent real fixed point x^* lies on the boundary of U.*

The following theorem characterizes the Julia set of $A_\lambda(z)$ for $\lambda = \lambda_A^*$.

Theorem 3.2.4. *Let $A_\lambda \in \mathcal{A}$ and $Esc(A_\lambda) = clo \ \{z \in \mathbb{C} : A_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$ be the closure of the set of escaping points of $A_\lambda(z)$. If $\lambda = \lambda_A^*$, then the Julia set $\mathcal{J}(A_\lambda) = Esc(A_\lambda)$.*

Proof. Let $z_0 \in \mathcal{J}(A_{\lambda_A^*})$ and V be any neighborhood of z_0. Since $\left\{A_{\lambda_A^*}^n\right\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\cup_n \left\{A_{\lambda_A^*}^n(V)\right\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > 0$ such that $\hat{x} \in \cup_n \left\{A_{\lambda_A^*}^n(V)\right\}$. Therefore, there exists $\hat{z} \in V$ such that $A_{\lambda_A^*}^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 3.2.2(b), it follows that $A_{\lambda_A^*}^n(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > 0$. Thus,

there exists a point $\hat{z} \in V$ such that $A^n_{\lambda^*_A}(\hat{z}) \to \infty$ as $n \to \infty$ and $z_0 \in Esc(A_{\lambda^*_A})$ follows. Consequently, $\mathcal{J}(A_{\lambda^*_A}) \subseteq Esc(A_{\lambda^*_A})$.

The proof for $Esc(A_{\lambda^*_A}) \subseteq \mathcal{J}(A_{\lambda^*_A})$ is similar to that in Theorem 3.2.3 for the case $0 < \lambda < \lambda^*_A$. □

Remark 3.2.4. *(i) The characterization of the Julia set $\mathcal{J}(A_\lambda)$ for $\lambda = \lambda^*_A$ found in Theorem 3.2.4 is useful in computationally generating the pictures of the Julia set in Section 3.4.*
(ii) Since $A^n_{\lambda^*_A}(x) \to \infty$ as $n \to \infty$ for $x > 0$ and by Theorem 3.2.4, it follows that the positive real points are contained in the Julia set of $A_\lambda(z)$ for $\lambda = \lambda^*_A$.

3.2.III Dynamics of $A_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda^*_A$

The dynamics $A_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda^*_A$, where λ^*_A is defined by (3.2.1), is described in this subsection. In this case also, we find the computationally useful characterization analogous to that of Theorems 3.2.3 and 3.2.4 for the Julia set $\mathcal{J}(A_\lambda)$.

We observe that unlike the case for $\lambda = \lambda^*_A$ in 3.2.II, wherein the point $x = 0$ is found to be rationally indifferent fixed point of $A_{\lambda^*_A}(z)$, in the present case, $\lambda > \lambda^*_A$, the point $x = 0$, in view of Theorem 3.2.1(c), turns out to be a repelling fixed point.

Theorem 3.2.5. *Let $A_\lambda \in \mathcal{A}$ and $Esc(A_\lambda) = clo\ \{z \in \mathbb{C} : A^n_\lambda(z) \to \infty \text{ as } n \to \infty\}$ be the closure of the set of escaping points of $A_\lambda(z)$. If $\lambda > \lambda^*_A$, then the Julia set $\mathcal{J}(A_\lambda) = Esc(A_\lambda)$.*

Proof. Let $z_0 \in \mathcal{J}(A_\lambda)$ and V be any neighborhood of z_0. Since $\{A^n_\lambda\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \{A^n_\lambda(V)\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > 0$ such that $\hat{x} \in \bigcup_n \{A^n_\lambda(V)\}$. Therefore, there exists $\hat{z} \in V$ such that $A^j_\lambda(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 3.2.2(c), it follows that $A^n_\lambda(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > 0$. Thus, there

exists a point $\hat{z} \in V$ such that $A_\lambda^n(\hat{z}) \to \infty$ as $n \to \infty$ so that $z_0 \in Esc(A_\lambda)$ follows. Consequently, $\mathcal{J}(A_\lambda) \subseteq Esc(A_\lambda)$ for $\lambda > \lambda_A^*$.

The proof for $Esc(A_\lambda) \subseteq \mathcal{J}(A_\lambda)$ for $\lambda > \lambda_A^*$ is similar to that in Theorem 3.2.3 for the case $0 < \lambda < \lambda_A^*$. □

Remark 3.2.5. *(i) The characterization of the Julia set $\mathcal{J}(A_\lambda)$ for $\lambda > \lambda^*$, found in Theorem 3.2.5, is useful for computationally generating pictures of the Julia sets in Section 3.4.*
(ii) If $\lambda > \lambda_A^*$, by Theorem 3.2.2(c) it follows that all the positive real points are escaping points and hence by Theorem 3.2.5 they belong to the Julia set of $A_\lambda(z)$.

3.3 Dynamics of slow growth entire function $B_\lambda \in \mathcal{B}$

In the present section, the dynamics of the entire transcendental function $B_\lambda \in \mathcal{B}$ is studied.

Firstly, the dynamics of $B_\lambda(x) = \lambda B(x)$ for $x \geq 0$, is investigated. The existence and nature of the fixed points of $B_\lambda(x)$ is found in Theorem 3.3.1. Theorem 3.3.2 describes the dynamics of $B_\lambda(x)$ on the positive real line.

Since all the coefficients of the Taylor series expansion of $B(z)$ are positive, the functions $B(x)$ and $B'(x)$ are positive valued, strictly increasing and continuous functions for $x \geq 0$. Therefore, the function $\phi(x) = B(x) - xB'(x)$ is strictly decreasing and continuous in the interval $[0, \infty)$. Consequently, since $B(0) = 1$ and $B(x_0) < x_0 B'(x_0)$ for some $x_0 \equiv x_0(B)$, there exists a unique $x^* \in (0, x_0)$ such that

$$\phi(x) \begin{cases} > 0 & \text{for } 0 < x < x^* \\ = 0 & \text{for } x = x^* \\ < 0 & \text{for } x^* < x < \infty \end{cases} \tag{3.3.1}$$

Throughout in this section, we denote

$$\lambda_B^* = \frac{1}{B'(x^*)} < 1 \tag{3.3.2}$$

where, x^* is the unique positive real root of the equation $\phi(x) = 0$.

The following theorem describes the nature of the fixed points of $B_\lambda(x)$ for $x \geq 0$.

Theorem 3.3.1. *Let $B_\lambda(x) = \lambda B(x)$ for $x \geq 0$. where $B(z)$ is defined by (3.1.4).*

(a) *If $0 < \lambda < \lambda^*_B$. $B_\lambda(x)$ has an attracting fixed point and a repelling fixed point.*

(b) *If $\lambda = \lambda^*_B$. $B_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^*$.*

(c) *If $\lambda > \lambda^*_B$, $B_\lambda(x)$ has no fixed points for $x \geq 0$.*

Proof. Define $g_\lambda(x) = B_\lambda(x) - x = \lambda B(x) - x$ for $x \geq 0$. The zeros of $g_\lambda(x)$ are fixed points of $B_\lambda(x)$. The following are some of the basic properties of $g_\lambda(x)$ needed in sequel:
(i) $g_\lambda(0) = \lambda > 0$, and the function $g_\lambda(x)$ is positive and strictly increasing for all sufficiently large values of x.
(ii) the function $g'_\lambda(x)$ is strictly increasing for $x \geq 0$ and $g'_\lambda(x) \to \infty$ as $x \to \infty$.
(iii) The function $g_\lambda(x)$ has a unique local minimum in $[0, \infty)$ for $0 < \lambda < 1/B'(0)$.
To see this, first we note that $B'(x)$ is strictly increasing positive valued function for $x \geq 0$ and $g'_\lambda(0) < 0$ for $0 < \lambda < 1/B'(0)$. Therefore, using the continuity of $g'_\lambda(x)$ and (ii), for $0 < \lambda < 1/B'(0)$ there exist a unique real number $\tilde{x} \equiv \tilde{x}(\lambda) > 0$ such that $g'_\lambda(\tilde{x}) = 0$, $g'_\lambda(x) < 0$ for $0 \leq x < \tilde{x}$ and $g'_\lambda(x) > 0$ for $x > \tilde{x}$. Thus, in view of $g''_\lambda(\tilde{x}) > 0$, $g_\lambda(x)$ attains a unique local minimum value at $x = \tilde{x}$ for $x \geq 0$ and $0 < \lambda < 1/B'(0)$.
(a) Clearly, $g'_\lambda(\tilde{x}) = 0$ implies that $\lambda = 1/B'(\tilde{x})$. Since $\lambda^*_B = 1/B'(x^*)$ and $B'(x)$ is strictly increasing function for $x \geq 0$, it follows that $\tilde{x} > x^*$ for $0 < \lambda < \lambda^*_B$. Thus, by (3.3.1), $B'(\tilde{x})\, g_\lambda(\tilde{x}) = B(\tilde{x}) - \tilde{x}B'(\tilde{x}) = \phi(\tilde{x}) < 0$. Consequently, $g_\lambda(\tilde{x}) < 0$ since $B'(\tilde{x}) > 0$. Now, (i), (iii) and $g_\lambda(\tilde{x}) < 0$ imply that $g_\lambda(x)$ has only two zeros a_λ and r_λ (say) with $a_\lambda < \tilde{x} < r_\lambda$. Since $a_\lambda < \tilde{x} < r_\lambda$ implies that $g'_\lambda(a_\lambda) < 0$ and $g'_\lambda(r_\lambda) > 0$, it follows that $B'_\lambda(a_\lambda) < 1$ and $B'_\lambda(r_\lambda) > 1$. Thus, the point a_λ is an attracting fixed point and the point r_λ is a repelling fixed point of $B_\lambda(x)$. This proves (a).

(b) If $\lambda = \lambda^*_B$, $\tilde{x} = x^*$, $g_\lambda(x^*) = 0$ and $g'_\lambda(x^*) = 0$. Consequently, $B'_{\lambda^*_B}(x^*) = 1$. Thus, in view of (iii), $B_{\lambda^*_B}(x)$ has a unique rationally indifferent fixed point at $x = x^*$.

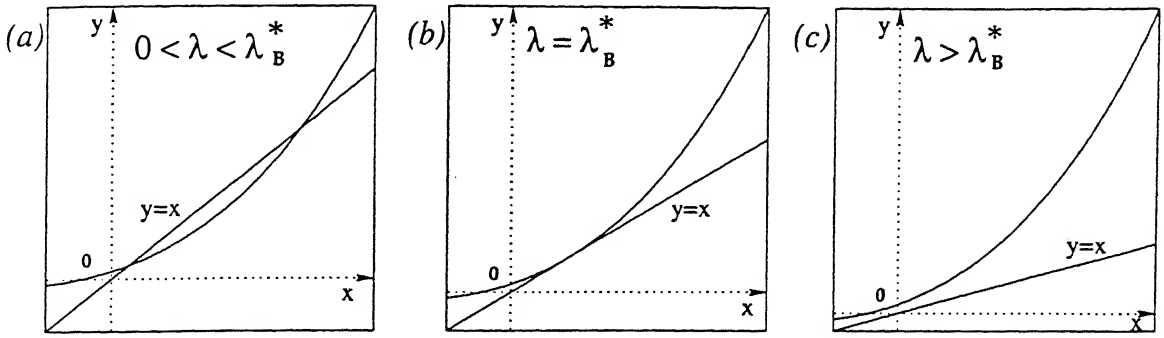


Figure 3.3: The graphs of $B_\lambda(x)$ for (a) $0 < \lambda < \lambda^*_B$, (b) $\lambda = \lambda^*_B$ and (c) $\lambda > \lambda^*_B$.

This proves (b).

(c) If $\lambda^*_B < \lambda < 1/B'(0)$ then, by (ii), $0 < \tilde{x} < x^*$ so that, in view of (3.1.1), $g_\lambda(\tilde{x}) > 0$. Consequently, by (iii), for $x \neq \tilde{x}$ $g_\lambda(x) > g_\lambda(\tilde{x}) > 0$ for all $x \geq 0$ and hence $g_\lambda(x) > 0$ for $x \geq 0$ and $\lambda^*_B < \lambda < 1/B'(0)$. Next, if $\lambda \geq 1/B'(0)$, by (ii) and the fact that $g'_\lambda(0) \geq 0$, $g'_\lambda(x) > 0$ for $x > 0$. Since $g_\lambda(0) > 0$, it follows that $g_\lambda(x) > 0$ for $x \geq 0$ and $\lambda \geq 1$. Thus, for all $x \geq 0$ and $\lambda > \lambda^*_B$, $g_\lambda(x) > 0$ so that $B_\lambda(x)$ has no fixed points for $x \geq 0$ and $\lambda > \lambda^*_B$. This completes the proof of (c). □

The dynamics of $B_\lambda(x)$ for $x \geq 0$ and $\lambda > 0$ is described with a phase portrait in the following theorem:

Theorem 3.3.2. *Let $B_\lambda(x) = \lambda B(x)$ for $x \geq 0$, where $B(z)$ is defined by (3.1.4).*

(a) *If $0 < \lambda < \lambda^*_B$, $B^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $0 \leq x < r_\lambda$ and $B^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$, where a_λ and r_λ are the attracting and the repelling fixed points of $B_\lambda(x)$ respectively.*

(b) *If $\lambda = \lambda^*_B$, $B^n_\lambda(x) \to x^*$ as $n \to \infty$ for $0 \leq x < x^*$ and $B^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x > x^*$, where x^*, given by (3.3.2), is the rationally indifferent fixed point of $B_\lambda(x)$.*

(c) *If $\lambda > \lambda^*_B$, $B^n_\lambda(x) \to \infty$ as $n \to \infty$ for all $x \geq 0$.*

Proof. The function $B'_\lambda(x)$ is strictly increasing for $x \geq 0$, $B'_\lambda(x) \to \infty$ as $x \to \infty$. Therefore, due to continuity of $B'_\lambda(x)$ and the fact that $B'_\lambda(0) < 1/B'(0)$ for $0 < \lambda < 1$,

there exist a unique real number $\tilde{x} \equiv \tilde{x}(\lambda)$ such that

$$B'_\lambda(\tilde{x}) \begin{cases} <1 & \text{for } 0 \le x < \tilde{x} \\ =1 & \text{for } x = \tilde{x} \\ >1 & \text{for } x > \tilde{x} \end{cases} \tag{3.3.3}$$

(a) If $0 < \lambda < \lambda_B^*$, by Theorem 3.3.1(a), it follows that $B_\lambda(x)$ has an attracting fixed point a_λ (say) and a repelling fixed point r_λ (say) with $0 < a_\lambda < \tilde{x} < r_\lambda$. Now, as in the proof of Theorem 3.3.1(a), $g_\lambda(x) = B_\lambda(x) - x$ has only two zeros at a_λ and r_λ, and $g_\lambda(\tilde{x}) < 0$ so that

$$B_\lambda(x) - x = g_\lambda(x) \begin{cases} >0 & \text{for } x \in [0, a_\lambda) \cup (r_\lambda, \infty) \\ <0 & \text{for } x \in (a_\lambda, r_\lambda) \end{cases} \tag{3.3.4}$$

Therefore, for $a_\lambda < x < r_\lambda$,

$$B_\lambda(x) - a_\lambda < x - a_\lambda . \tag{3.3.5}$$

If $0 \le x < a_\lambda$ then, by the mean value theorem, $|B_\lambda(x) - a_\lambda| \le B'_\lambda(c)|x - a_\lambda|$, where $x < c < a_\lambda$. Since, $c < \tilde{x}$, $B'_\lambda(x)$ is strictly increasing for $x \ge 0$ and by (3.3.3), it follows that $B'_\lambda(c) < 1$. Consequently, $|B_\lambda(x) - a_\lambda| < |x - a_\lambda|$ for $0 \le x < a_\lambda$. This inequality together with inequality (3.3.5) gives that for $0 \le x < r_\lambda$ and $x \ne a_\lambda$, $|B_\lambda(x) - B_\lambda(a_\lambda)| = |B_\lambda(x) - a_\lambda| < |x - a_\lambda|$. Thus, for $0 \le x < r_\lambda$, $B_\lambda^n(x) \to a_\lambda$ as $n \to \infty$. Further, if $x > r_\lambda$, by (3.3.3) and (3.3.4), $B_\lambda^n(x) \to \infty$ as $n \to \infty$. This completes the proof of (a).

(b) By Theorem 3.3.1(b), if $\lambda = \lambda_B^*$ then $B_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^*$. Since $B'_{\lambda_B^*}(x^*) = 1$ and $B'_{\lambda_B^*}(x)$ is strictly increasing for $x \ge 0$, $B'_{\lambda_B^*}(x) < 1$ for $0 \le x < x^*$, and $B'_{\lambda_B^*}(x) > 1$ for $x > x^*$. Now, as in proof of (a) above, it follows that if $0 \le x < x^*$, $|B_{\lambda_B^*}(x) - x^*| < |x - x^*|$. Therefore, $B^n_{\lambda_B^*}(x) \to x^*$ as $n \to \infty$, for $0 \le x < x^*$. If $x > x^*$ then $B^n_{\lambda_B^*}(x) \to \infty$ as $n \to \infty$, since $B_{\lambda_B^*}(x) > x$ for $x > x^*$. This proves (b).

(c) If $\lambda > \lambda_B^*$, by the arguments in the proof of Theorem 3.3.1(c) it follows that $g_\lambda(x) = B_\lambda(x) - x > 0$ for $x \ge 0$ and hence $B_\lambda^n(x) \to \infty$ as $n \to \infty$, completing the proof of (c).

□

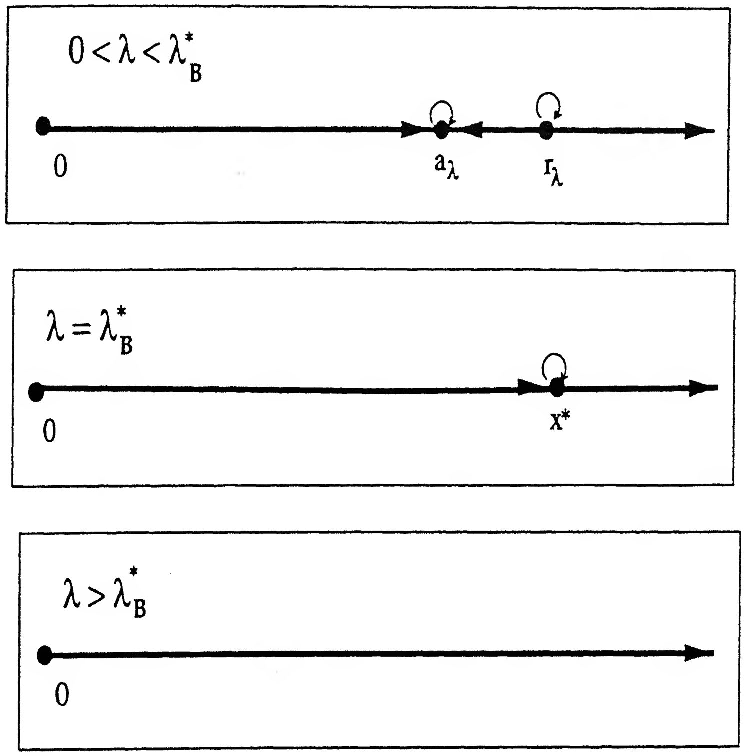


Figure 3.4: Phase portrait of the function $B_\lambda(x) = \lambda B(x)$ for $x \geq 0$ and $\lambda > 0$.

3.3.I Dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda_B^*$

This subsection is devoted to the investigation of the dynamics $B_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda_B^*$, where λ_B^* is defined by (3.3.2).

If $0 < \lambda < \lambda_B^*$, by Theorem 3.3.1(a), $B_\lambda(z)$ has a real attracting fixed point a_λ . Let $\mathsf{A}(a_\lambda)$ be the basin of attraction of a_λ *i.e.*

$$\mathsf{A}(a_\lambda) = \{z \in \mathbb{C} : B_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\} .$$

In Proposition 3.3.1, it is shown that the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point a_λ of $B_\lambda(z)$, $0 < \lambda < \lambda_B^*$ contains the points which get mapped by B_λ inside the open disk $D_{\tilde{x}}(0)$ centered at origin and having radius $\tilde{x}$, where $\tilde{x}$ is the positive real number such that $B_\lambda'(\tilde{x}) = 1$. Theorem 3.3.3 gives computationally useful characterization of the Julia set $\mathcal{J}(B_\lambda)$ as the closure of the set of escaping points of $B_\lambda(z)$.

Proposition 3.3.1. *Let $B_\lambda \in \mathcal{B}$ and $0 < \lambda < \lambda_B^*$. Then, the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point a_λ of $B_\lambda(z)$ contains the set $D = \{z \in \mathbb{C} : |B_\lambda(z)| < \tilde{x}\}$. where $\tilde{x}$ is the real number such that $B_\lambda'(\tilde{x}) = 1$. Further, $\tilde{x} > x^*$, where x^* is given by (3.3.2).*

Proof. Let $B_\lambda(z) = \lambda B(z)$ for $z \in \mathbb{C}$. Since $\frac{1}{B'(\tilde{x})} = \lambda < \lambda_B^* = \frac{1}{B'(x^*)}$ and $B'(x)$ is strictly increasing function for $x \geq 0$, it follows that $\tilde{x} > x^*$.

We first show that $B_\lambda(z)$ maps the open disk $D_{\tilde{x}}(0)$, centered at origin and having radius $\tilde{x}$ into itself. Since $B'(x) \geq 0$ for all $x \in [0, \infty)$, by (3.3.1), $B'(\tilde{x})\left(\frac{B(\tilde{x})}{B'(\tilde{x})} - \tilde{x}\right) = B(\tilde{x}) - \tilde{x}B'(\tilde{x}) = \phi(\tilde{x}) < 0$, it follows that $\frac{B(\tilde{x})}{B'(\tilde{x})} < \tilde{x}$. Consequently, $B_\lambda(x) = \lambda B(x) = \frac{1}{B'(\tilde{x})}B(x) < \frac{B(\tilde{x})}{B'(\tilde{x})} < \tilde{x}$, for $x < \tilde{x}$. Therefore, $\max_{|z|=\tilde{x}} |B_\lambda(z)| \leq B_\lambda(\tilde{x}) \leq \frac{B(\tilde{x})}{B'(\tilde{x})} < \tilde{x}$, since $|B_\lambda(z)| \leq B_\lambda(|z|)$. Now, the maximum modulus principle gives $|B_\lambda(z)| < \tilde{x}$ for all $z \in D_{\tilde{x}}(0)$. Thus, $B_\lambda(z)$ maps the open disk $D_{\tilde{x}}(0)$ into itself.

Since $D_{\tilde{x}}(0)$ is a simply connected domain and $B_\lambda(D_{\tilde{x}}(0)) \subseteq D_{\tilde{x}}(0)$, by Schwarz lemma ([30], p264), $B_\lambda^n(z) \to a_\lambda$ as $n \to \infty$ for all $z \in D_{\tilde{x}}(0)$. It is easily seen from the definition of D that $B_\lambda(z)$ maps D into $D_{\tilde{x}}(0)$ and hence $B_\lambda^n(z) \to a_\lambda$ as $n \to \infty$ for all $z \in D$. Thus, $\mathsf{A}(a_\lambda) \supset D$. $\square$

Remark 3.3.1. *(i) By Theorem 3.3.2(a), $B_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $0 \leq x < r_\lambda$, where r_λ is the repelling fixed point of $B_\lambda(z)$ and $0 < \lambda < \lambda_B^*$. Therefore, the basin of attraction $\mathsf{A}(a_\lambda)$ of the attracting fixed point a_λ contains the interval $[0, r_\lambda)$ for $0 < \lambda < \lambda_B^*$.*

(ii) As in the proof of Proposition 3.3.1, the function $B_\lambda(z)$ maps the disk $D_{\tilde{x}}(0)$ centered at origin and having radius $\tilde{x}$ into itself when $0 < \lambda < \lambda_B^*$. Therefore, $D_{\tilde{x}}(0) \subset D \subseteq \mathsf{A}(a_\lambda) \subseteq \mathcal{F}(B_\lambda)$.

By Theorem 3.3.1(a), for $0 < \lambda < \lambda_B^*$, r_λ is the repelling fixed point for $B_\lambda(z)$ and therefore r_λ belongs to the Julia set $\mathcal{J}(B_\lambda)$ of $B_\lambda(z)$. In the following theorem, we find

that the Julia set $\mathcal{J}(A_\lambda)$ is the closure of the set of escaping points for $0 < \lambda < \lambda_B^*$.

Theorem 3.3.3. *Let $B_\lambda \in \mathcal{B}$ and $Esc(B_\lambda) = clo\ \{z \in \mathbb{C} : B_\lambda^n(z) \to \infty$ as $n \to \infty\}$ be the closure of the set of escaping points of $B_\lambda(z)$. If $0 < \lambda < \lambda_B^*$, then the Julia set $\mathcal{J}(B_\lambda) = Esc(B_\lambda)$.*

Proof. Let $z_0 \in \mathcal{J}(B_\lambda)$ and U be any neighborhood of z_0. Since $\{B_\lambda^n\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \{B_\lambda^n(U)\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > r_\lambda$ such that $\hat{x} \in \bigcup_n \{B_\lambda^n(U)\}$, where r_λ is the repelling fixed point of $B_\lambda(z)$ for $0 < \lambda < \lambda_B^*$. Therefore, there exists a point $\hat{z} \in U$ such that $B_\lambda^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 3.3.2(a), it follows that $B_\lambda^n(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > r_\lambda$. Thus, there exists a point $\hat{z} \in U$ such that $B_\lambda^n(\hat{z}) \to \infty$ as $n \to \infty$ and $z_0 \in Esc(B_\lambda)$ follows. Consequently, $\mathcal{J}(B_\lambda) \subseteq Esc(B_\lambda)$.

To prove the reverse containment relation, let $z_1 \in Esc(B_\lambda)$. Without loss of generality it is assumed that $B_\lambda^n(z_1) \to \infty$ as $n \to \infty$. Suppose $z_1 \notin \mathcal{J}(B_\lambda)$. Then, there exists a neighborhood V of z_1 such that $\{B_\lambda^n\}$ is normal in V. This means that $\{B_\lambda^n\}$ contains either (i) a subsequence which converges to a limit function $g \not\equiv \infty$ uniformly on each compact subset of V, or (ii) a subsequence which converges to ∞ on each compact subset of V.

Case (i): Let $B_\lambda^{n_k}(z) \to g(z)$ for all $z \in V$ as $n \to \infty$. By Weirstrass theorem, the function $g(z)$ is analytic in V. Let $w \in V$. Then, it must lie in some component of $\mathcal{F}(B_\lambda)$. Since the generalized $(\log, \log)$- order of $B_\lambda(z)$ is $1+\mu$, where $1 < \mu < 2$, by Theorem 1.3.8, every component of the Fatou set $\mathcal{F}(B_\lambda)$ is bounded. It therefore follows that $|B^n(w)| < M$, $n = 1, 2, \cdots$, for some M and hence $|g(w)| < M$. But, the forward orbit of the point $z_1 \in V$ escapes to ∞ under iteration of B_λ. Thus, as $n_k \to \infty$, $B_\lambda^{n_k}(z_1) \to \infty$, while $B_\lambda^{n_k}(w) \to g(w)$ with $|g(w)| < M$ for $w \in V$. This implies that $\{B_\lambda^n\}$ is not normal at z_1.

Case (ii) If $B_\lambda^{n_k}(z) \to \infty$ for all $z \in V$ as $n_k \to \infty$, then for each M^* sufficiently large and

any $z^* \in V$, $|B_\lambda^N(z^*)| > M^*$ for some positive integer $N \equiv N(z^*, M^*)$. Since $V \subset \mathcal{F}(B_\lambda)$, by the invariance property of the Fatou set, the point $B_\lambda^N(z^*)$ must lie in some component of the Fatou set of $B_\lambda(z)$. It therefore follows that atleast one component of the Fatou set $\mathcal{F}(B_\lambda)$ is unbounded. This contradicts the fact that every component of the Fatou set $\mathcal{F}(B_\lambda)$ is bounded (c.f. Theorem 1.3.8). Therefore, $\{B_\lambda^n\}$ is not normal at z_1.

By case (I) and case (II) it follows that $z_1 \in \mathcal{J}(B_\lambda)$, a contradiction. Thus, $Esc(B_\lambda) \subseteq \mathcal{J}(B_\lambda)$. □

Remark 3.3.2. *(i) Theorem 3.3.3 provides a characterization of the Julia set of $B_\lambda(z)$ as the closure of the set of all escaping points of $B_\lambda(z)$ and it is quite useful in computationally generating the pictures of the Julia set of $B_\lambda(z)$ in Section 3.4.*

(ii) By Theorem 3.3.2(a), $B_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$, where r_λ is the repelling fixed point of $B_\lambda(z)$, $0 < \lambda < \lambda_B^*$. Therefore, by Theorem 3.3.3 it follows that the interval (r_λ, ∞) is contained in the Julia set of $B_\lambda(z)$ for $0 < \lambda < \lambda_B^*$.

3.3.II Dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda = \lambda_B^*$

In this subsection, the dynamics $B_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda = \lambda_B^*$ is described, where λ_B^* is defined by (3.3.2).

We first show in Proposition 3.3.2 that if $\lambda = \lambda_B^*$, the Fatou set of $B_\lambda(z)$ contains a parabolic domain U corresponding to the rationally indifferent fixed point x^*. A general description of the parabolic domain U is found in Proposition 3.3.2. The computationally useful characterization similar to Theorem 3.3.3 of the Julia set $\mathcal{J}(B_\lambda)$, $\lambda = \lambda_B^*$ is obtained in Theorem 3.3.4.

Proposition 3.3.2. *Let $B_\lambda \in \mathcal{B}$ and $\lambda = \lambda_B^*$. Then, the Fatou set $\mathcal{F}(B_\lambda)$ contains a parabolic domain.*

Proof. Let $\mathsf{U} = \left\{z \in \mathbb{C} : B_{\lambda_B^*}^n(z) \to x^* \text{ as } n \to \infty\right\}$, where x^* is given by (3.3.2). By Theorem 3.3.1(b), it follows that $B_\lambda(z)$ has a rationally indifferent fixed point at $x = x^*$.

Since, by Theorem 3.3.2(b), as $n \to \infty$, $B^n_{\lambda^*_B}(x) \to x^*$ for $0 \le x < x^*$ and $B^n_{\lambda^*_B}(x) \to \infty$ for $x > x^*$, the rationally indifferent fixed point x^* lies on the boundary of U. Thus, U is a parabolic domain in the Fatou set of $B_{\lambda^*_B}(z)$. $\square$

Remark 3.3.3. *(i) As in the proof of Proposition 3.3.2, $B^n_\lambda(x) \to x^*$ for $0 < x < x^*$ and $B^n_\lambda(x) \to \infty$ for $x > x^*$ as $n \to \infty$, when $\lambda = \lambda^*_B$. Therefore, in this case, the indifferent fixed point x^* belongs to the Julia set of $B_{\lambda^*_B}(z)$ and the interval $(0, x^*)$ is contained in the parabolic domain* U.

(ii) It follows from the proof of Proposition 3.3.2 that the rationally indifferent real fixed point x^* lies on the boundary of U.

The following proposition shows that, for $\lambda = \lambda^*_B$, the set of all points which get mapped inside the open disk $D_{x^*}(0)$ centered at 0 and having radius x^* is contained in the parabolic domain U.

Proposition 3.3.3. *Let $B_\lambda \in \mathcal{B}$ and $\lambda = \lambda^*_B$. Then, the parabolic domain $\mathsf{U} = \{z : B^n_\lambda \to x^* \text{ as } n \to \infty\}$ corresponding to the rationally indifferent real fixed point x^* contains the set $D = \{z \in \mathbb{C} : |B_\lambda(z)| < x^*\}$, where x^* is given by (3.3.2).*

Proof. Since x^* is the rationally indifferent fixed point, $B'_{\lambda^*_B}(x^*) = 1$. We first show that $B_{\lambda^*_B}(z)$ maps the open disk $D_{x^*}(0)$, centered at origin and having radius x^* into itself. Since, by (3.3.1), $B'(x^*)\left(\frac{B(x^*)}{B'(x^*)} - x^*\right) = B(x^*) - x^*B'(x^*) = \phi(x^*) = 0$, it follows that $\frac{B(x^*)}{B'(x^*)} = x^*$. Consequently, $B_{\lambda^*_B}(x) = \lambda^*_B B(x) = \frac{1}{B'(x^*)}B(x) < \frac{B(x^*)}{B'(x^*)} = x^*$, for $0 \le x < x^*$. Therefore, $\max_{|z|=x^*} \left|B_{\lambda^*_B}(z)\right| \le B_{\lambda^*_B}(x^*) \le \frac{B(x^*)}{B'(x^*)} = x^*$, since $|B_\lambda(z)| \le B_\lambda(|z|)$. Now, the maximum modulus principle gives $\left|B_{\lambda^*_B}(z)\right| < x^*$ for $z \in D_{x^*}(0)$. Thus, $B_{\lambda^*_B}(z)$ maps the open disk $D_{x^*}(0)$ into itself.

Since $D_{x^*}(0)$ is a simply connected domain and $B_{\lambda^*_B}(D_{x^*}(0)) \subseteq D_{x^*}(0)$, $B^n_{\lambda^*_B}(z) \to x^*$ as $n \to \infty$ for all $z \in D_{x^*}(0)$. Since $B_{\lambda^*_B}(z)$ maps D into $D_{x^*}(0)$ and $B^n_{\lambda^*_B}(z) \to x^*$ as

$n \to \infty$ for all $z \in D_{x^*}(0)$, it follows that $B^n_{\lambda^*_B}(z) \to x^*$ as $n \to \infty$ for all $z \in D$. Thus, $\mathrm{U} \supset D$. □

Remark 3.3.4. *By the arguments in the proof of Proposition 3.3.3, it follows that $B_{\lambda^*_B}(z)$ maps the open disk $D_{x^*}(0)$, centered at origin and having radius x^* into itself. Therefore,*

$$D_{x^*}(0) \subseteq D \subseteq \mathrm{U} \subseteq \mathcal{F}(B_{\lambda^*_B}) \ .$$

By Remark 3.3.3(i), for $\lambda = \lambda^*_B$ the rationally indifferent fixed point x^* belongs to the Julia set $\mathcal{J}(B_\lambda)$. In this case, the following theorem leads to the assertion that the interval (x^*, ∞) is contained in the Julia set $\mathcal{J}(B_\lambda)$.

Theorem 3.3.4. *Let $B_\lambda \in \mathcal{B}$ and $Esc(B_\lambda) = clo\ \{z \in \mathbb{C} : B^n_\lambda(z) \to \infty \text{ as } n \to \infty\}$ be the closure of the set of escaping points of $B_\lambda(z)$. If $\lambda = \lambda^*_B$, then the Julia set $\mathcal{J}(B_\lambda) = Esc(B_\lambda)$.*

Proof. Let $z_0 \in \mathcal{J}(B_{\lambda^*_B})$ and V be any neighborhood of z_0. Since $\left\{B^n_{\lambda^*_B}\right\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \left\{B^n_{\lambda^*_B}(V)\right\}$ omits at most one point in $\mathbb{C}$, where x^* is the rationally indifferent fixed point of $B_{\lambda^*_B}$. In particular, there is a point $\hat{x} > x^*$ such that $\hat{x} \in \bigcup_n \left\{B^n_{\lambda^*_B}(V)\right\}$. Therefore, there exists a point $\hat{z} \in V$ such that $B^j_{\lambda^*_B}(\hat{z}) = \hat{x}$ for some positive integer j. Now, since $\hat{x} > x^*$, by Theorem 3.3.2(b), it follows that $B^n_{\lambda^*_B}(\hat{x}) \to \infty$ as $n \to \infty$. Thus, there exists a point $\hat{z} \in V$ such that $B^n_{\lambda^*_B}(\hat{z}) \to \infty$ as $n \to \infty$ and consequently, $z_0 \in Esc(B_{\lambda^*_B})$, proving the containment relation $\mathcal{J}(B_{\lambda^*_B}) \subseteq Esc(B_{\lambda^*_B})$.

The proof of $Esc(B_\lambda) \subseteq \mathcal{J}(B_\lambda)$ for $\lambda = \lambda^*_B$ is similar to that in Theorem 3.3.3 for the case $0 < \lambda < \lambda^*_B$. □

Remark 3.3.5. *(i) Theorem 3.3.4 gives the computationally useful characterization of the Julia set $\mathcal{J}(B_\lambda)$ for $\lambda = \lambda^*_B$ similar to Theorem 3.3.3.*
(ii) Theorem 3.3.2(b) gives that all the points in the interval (x^*, ∞) are escaping points and hence, by Theorem 3.3.4 $(x^*, \infty) \subseteq \mathcal{J}(B_{\lambda^*_B})$ follows.

3.3.III Dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda_B^*$

The dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda_B^*$ is described in the present subsection, where λ_B^* is defined by (3.3.2). In this case also, the computationally useful characterization of the Julia set $\mathcal{J}(B_\lambda)$ is obtained.

Theorem 3.3.5. *Let $B_\lambda \in \mathcal{B}$ and $Esc(B_\lambda) = clo\ \{z \in \mathbb{C} : B_\lambda^n(z) \to \infty$ as $n \to \infty\}$ be the closure of the set of escaping points of $B_\lambda(z)$. If $\lambda > \lambda_B^*$, then the Julia set $\mathcal{J}(B_\lambda) = Esc(B_\lambda)$.*

Proof. Let $z_0 \in \mathcal{J}(B_\lambda)$ and V be any neighborhood of z_0. Since $\{B_\lambda^n\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \{B_\lambda^n(V)\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > 0$ such that $\hat{x} \in \bigcup_n \{B_\lambda^n(V)\}$. Therefore, there exists $\hat{z} \in V$ such that $B_\lambda^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 3.3.2(c), it follows that $B_\lambda^n(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > 0$. Thus, there exists a point $\hat{z} \in V$ such that $B_\lambda^n(\hat{z}) \to \infty$ as $n \to \infty$ so that $z_0 \in Esc(B_\lambda)$ follows. Consequently, $\mathcal{J}(B_\lambda) \subseteq Esc(B_\lambda)$.

The proof for $Esc(B_\lambda) \subseteq \mathcal{J}(B_\lambda)$ for $\lambda > \lambda_B^*$ is similar to that in Theorem 3.3.3 for the case $0 < \lambda < \lambda_B^*$. □

Remark 3.3.6. *(i) The characterization of the Julia set $\mathcal{J}(B_\lambda)$ for $\lambda > \lambda_B^*$ obtained in Theorem 3.3.5 is useful in computationally generating the pictures of the Julia set $\mathcal{J}(B_\lambda)$ for $\lambda > \lambda_B^*$ in Section 3.4.*
(ii) If $\lambda > \lambda_B^*$, by Theorem 3.3.2(c), $B_\lambda^n(x) \to \infty$ for $x \geq 0$ as $n \to \infty$. Therefore, by the above theorem it follows that the non-negative real points are contained in the Julia set of $B_\lambda(z)$ for $\lambda > \lambda_B^*$

3.4 Applications

The characterization of the Julia set $\mathcal{J}(A_\lambda)$ obtained in Theorems 3.2.3, 3.2.4, and 3.2.5 and the characterization of the Julia set $\mathcal{J}(B_\lambda)$ obtained in Theorems 3.3.3, 3.3.4, and 3.3.5 give a useful algorithm to computationally generate the pictures of the Julia set of $A_\lambda(z)$ and $B_\lambda(z)$. The algorithm runs as follows:

1. *Select a rectangular domain R in the plane and construct a $m \times n$ grid in this rectangle.*
2. *For each grid point, compute the orbit of this point up to a maximum of N iterations.*
3. *If, at iteration $i < N$, the absolute value of the orbit is greater than some given bound M, the original grid point is colored black and the iterations are stopped for that grid point.*
4. *If absolute value of no point in the orbit ever becomes greater than M, the original grid point is left as white.*

Thus, in the output generated by this algorithm, the black points represent the Julia set of the function and the white points represent the Fatou set of the function. Sometimes, either the absolute value of the orbits of certain white points may take longer than N iterations to escape the bound M or the absolute value of the orbits of certain black points may take longer than N iterations to become smaller than the given bound M. These aspects depend on the choice of N and M.

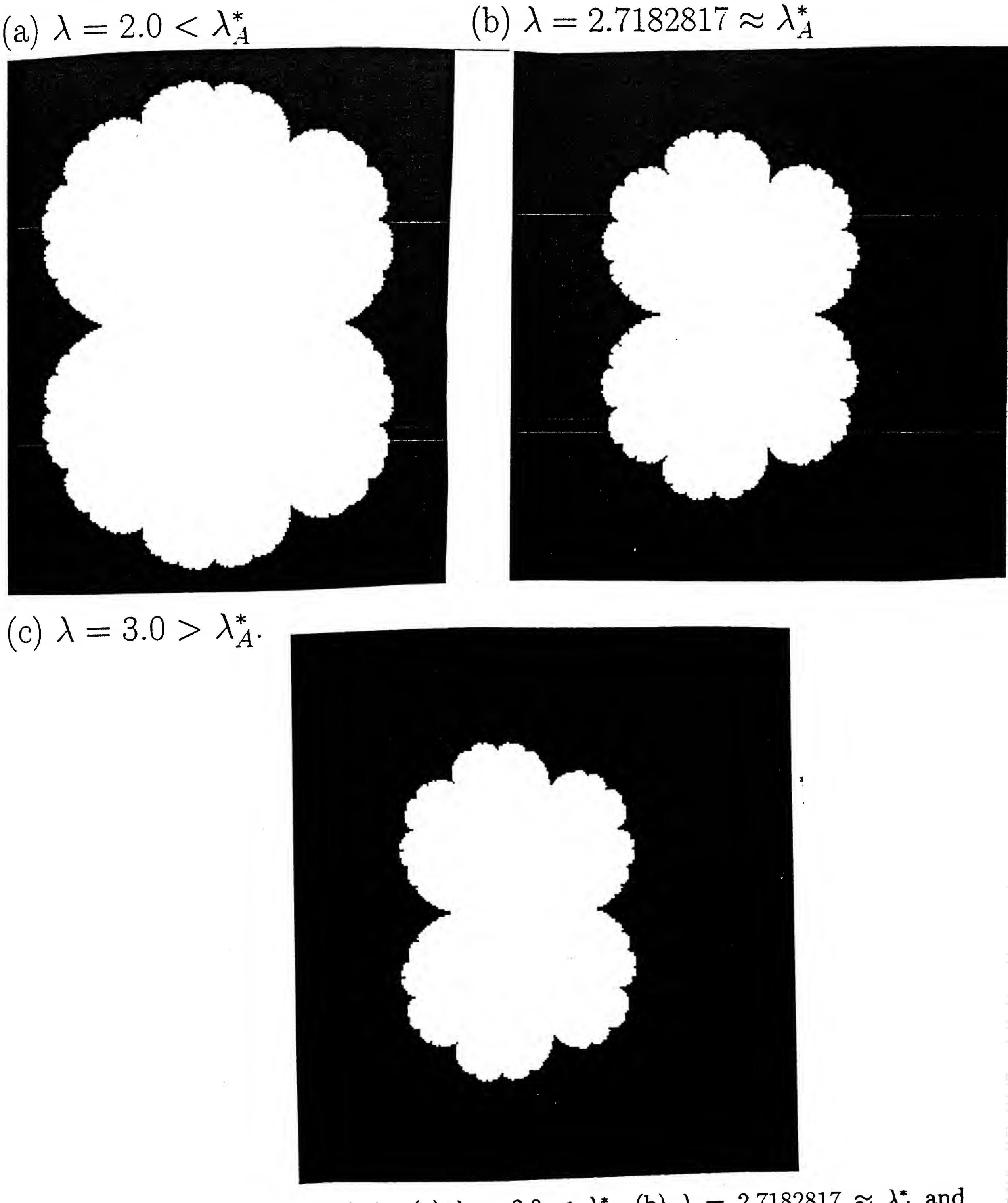
Consider the general T-fraction

$$C(z) \equiv \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{z/\left(n^2 \exp\left(n^{\frac{1}{\mu}}\right)\right)}{1 + z/\left(n^2 \exp\left(n^{\frac{1}{\mu}}\right)\right)} \right), \qquad \mu = 1.5 \tag{3.4.1}$$

Then, the sequence of numerators $\{A_n(z)\}$ and denominators $\{B_n(z)\}$ of the approximants of the continued fraction (3.4.1) converge uniformly to the entire functions $A(z)$ and $B(z)$ respectively, on each compact subsets of $\mathbb{C}$. The pictures of the Julia sets of $A_\lambda(z) = \lambda A(z)$ and $B_\lambda(z) = \lambda B(z)$ for different values of λ are computationally generated based on the above algorithm.

For the functions in $A_\lambda(z)$, it is found that $\lambda_A^* \approx 2.7182817$ by using (3.2.1). Let $R = \{z \in \mathbb{C} : -30 \le \Re(z) \le 20 \text{ and } -30 \le \Im(z) \le 30\}$, the maximum number of iterations $N = 75$ and $M = 100$. The resulting pictures of the Julia set of $A_\lambda(z)$ for $\lambda = 2.0 < \lambda_A^*$, $\lambda = 2.7182817 \approx \lambda_A^*$ and $\lambda = 3.0 > \lambda_A^*$ are shown in Figure 3.5.

For the functions in $B_\lambda(z)$, by (3.3.2), the value of λ_B^* is computed numerically by using bisection method and it is found that $\lambda_B^* \approx 1.07107$. Let $R = \{z \in \mathbb{C} : -90 \le \Re(z) \le 30 \text{ and } -60 \le \Im(z) \le 60\}$, the maximum number of iterations $N = 100$ and $M = 200$. The resulting pictures of the Julia set of $B_\lambda(z)$ for $\lambda = 1.0 < \lambda_B^*$, $\lambda = 1.07107 \approx \lambda_B^*$ and $\lambda = 1.5 > \lambda_B^*$ are shown in Figure 3.6.



3.5: Julia sets of $A_\lambda(z)$ for (a) $\lambda = 2.0 < \lambda_A^*$, (b) $\lambda = 2.7182817 \approx \lambda_A^*$ and $3.0 > \lambda_A^*$.

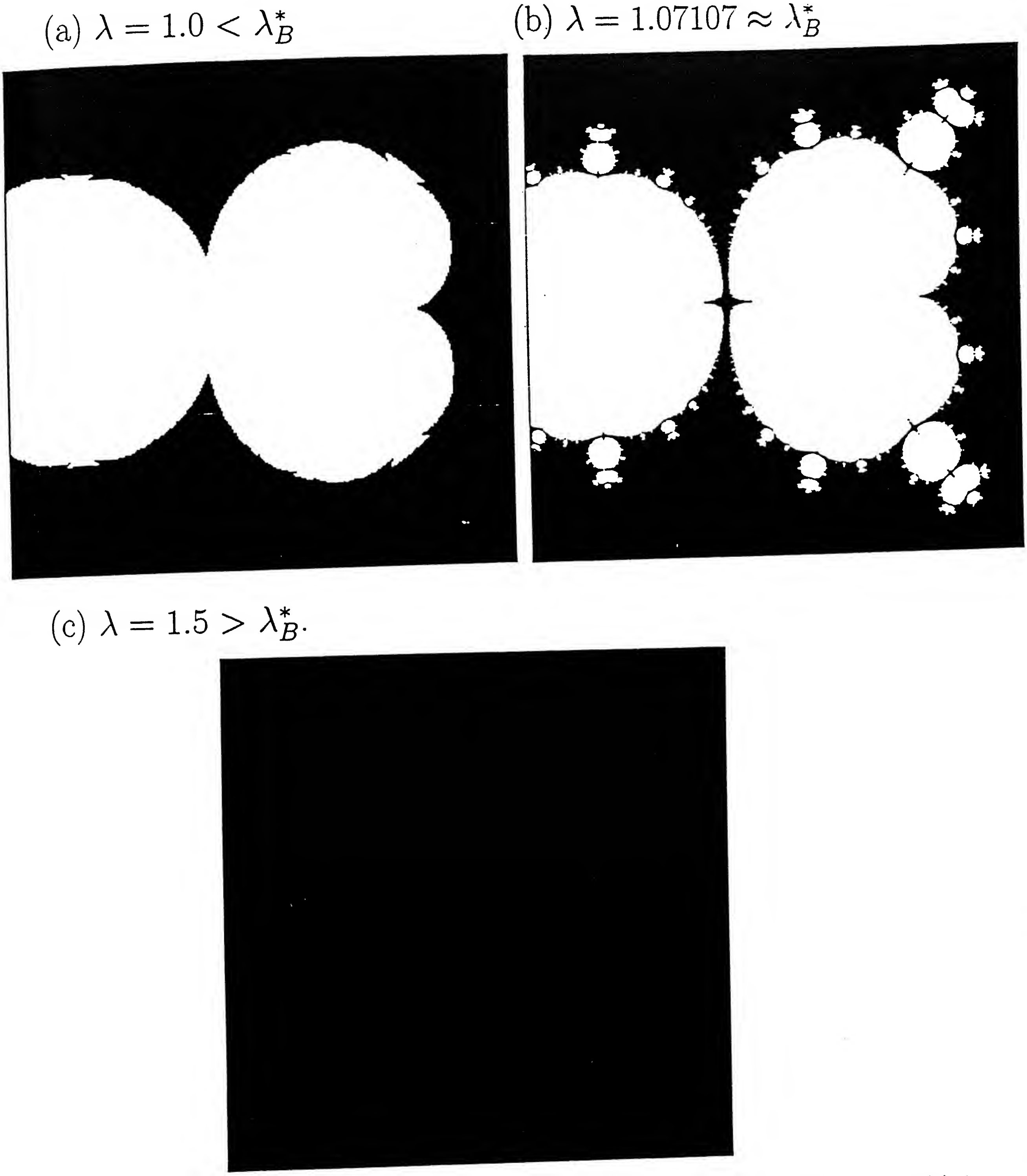


Figure 3.6: Julia sets of $B_\lambda(z)$ for (a) $\lambda = 1.0 < \lambda_B^*$, (b) $\lambda = 1.0717 \approx \lambda_B^*$ and (c) $\lambda = 1.5 > \lambda_B^*$.

Chapter 4

Dynamics of the non-critically finite entire function $(e^z - 1)/z$

In complex analytic dynamics, the singular values (c.f. Definition 1.1.12) of a function play an important role in determining the dynamics of a function. The dynamics of polynomials and dynamics of certain classes of transcendental entire functions are hitherto studied by taking advantage of the presence of only finitely many critical values and asymptotic values of their functions. The dynamical behaviour of critically finite (c.f. Definition 1.1.13) entire transcendental functions share many of the properties of polynomials and rational functions; for instance, these functions do not have wandering domains (c.f. Theorem 1.1.14). Exploiting the critical finiteness, Devaney and coworkers [26,31,33,36,37] studied exhaustively the dynamics of some of the most interesting periodic entire transcendental functions like λe^z, $\lambda \sin z$ and $\lambda \cos z$. However, the dynamics of non-critically finite entire functions has not been explored so far, probably because of non-applicability of Sullivan's theorem (c.f. Theorem 1.1.6) to these functions. Also, the presence of infinitely many critical values and the behaviour of the orbits of critical values make it difficult to study the dynamics of non-critically finite entire functions. In the present chapter an effort is made in this direction by studying the dynamics of the non-critically finite entire function $(e^z - 1)/z$ that arises as the denominator of a separately convergent general T-fraction. It is to be observed that besides being non-critically finite, the functions considered in

this chapter are of order (c.f. Definition 1.3.1) one, whereas the functions considered in Chapter 3 were of order zero.

4.1 One parameter family $\mathcal{K} \equiv \{f_\lambda(z) = \frac{\lambda(e^z-1)}{z} : \lambda > 0\}$

Let

$$C(z) \equiv \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{-z/(n+1)}{1+(z/(n+1))} \right) \tag{4.1.1}$$

be a general T-fraction.

Let $B_n(z)$ be the denominator of the nth approximant of the continued fraction (4.1.1). The function $B_n(z)$ satisfies the following three term recurrence relation

$$B_n(z) = \frac{z}{n+1} B_{n-1}(z) - \frac{z}{n+1} B_{n-2}(z) + B_{n-1}(z)$$

with initial conditions $B_{-1} \equiv 0$ and $B_0 \equiv 1$. It is easily seen from the above recurrence relation that the function $B_n(z)$ is represented in the form $B_n(z) = 1 + \sum_{k=1}^{n} \frac{z^k}{F_1 F_2 \cdots F_k}$ with $F_k = \frac{1}{k+1}$, $k = 1, 2, \ldots$. Therefore,

$$\lim_{n\to\infty} B_n(z) = \lim_{n\to\infty} \left(1 + \sum_{k=1}^{n} \frac{z^k}{F_1 F_2 \cdots F_k}\right) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{(k+1)!} \equiv \frac{e^z - 1}{z}$$

Here and in the sequel the function $(e^z - 1)/z$ is defined to be equal to 1 at $z = 0$. Thus, the sequence $\{B_n(z)\}_{n=1}^{\infty}$ converges to the entire function $B(z) = (e^z - 1)/z$ for all $z \in \mathbb{C}$. We observe that for the general T-fraction (4.1.1), giving rise to the entire function $(e^z - 1)/z$, $\sum |F_n| = \sum |G_n| = \sum \frac{1}{n+1} = \infty$, while the sequence of elements in the general T-fraction (3.1.1) that gave rise to the functions $A(z)$ and $B(z)$ in Chapter 3 formed convergent series.

The numerator $A_n(z)$ of the n th approximant satisfies the following three term recurrence relation

$$A_n(z) = \frac{z}{n+1} A_{n-1}(z) - \frac{z}{n+1} A_{n-2}(z) + A_{n-1}(z)$$

with initial condition $A_{-1} \equiv 1$ and $A_0 \equiv 0$. Since $A_n(z) = 1 - B_n(z)$, the convergence of $\{A_n(z)\}$ to $A(z) = 1 - B(z)$ follows by the convergence of $\{B_n(z)\}$. The dynamics of the

function $A(z)$ is derivable from the dynamics of the function $B(z)$, and vice versa. Thus, in the present chapter, without loss of generality, the dynamics of the denominator $B(z)$ is investigated.

Let

$$\mathcal{K} \equiv \left\{ f_\lambda(z) = \lambda \frac{e^z - 1}{z} : \lambda > 0 \right\}$$

be one parameter family of entire transcendental functions. The present chapter describes the dynamical behaviour of the entire transcendental non-critically finite functions $f_\lambda \in \mathcal{K}$. In Section 4.2, we develop some of the basic properties of $f_\lambda \in \mathcal{K}$ that are needed in the sequel. In particular, it is found in this section that $f_\lambda(z)$ has infinitely many critical values in the disk centered at origin and having radius λ and $f'_\lambda(z)$ has infinitely many zeros in the left half plane. In Section 4.3, the dynamics of $f_\lambda(x)$ for x belonging to the real line $\mathbb{R}$ and $\lambda > 0$ is described in detail. In this section, it is shown that there exists a critical parameter value $\lambda^* > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$ occurs at $\lambda = \lambda^*$ (≈ 0.64761). *i.e.*, if the parameter value crosses the value λ^*, then a dramatic change in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$, occurs. The dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda^*$ is studied in Section 4.4. For this case, we prove two different characterizations for the Julia set of $f_\lambda(z)$. The first characterization gives the Julia set $\mathcal{J}(f_\lambda)$, $0 < \lambda < \lambda^*$, as the closure of the set of escaping points; while the second characterization, describes it as the complement of the basin of attraction of an attracting real fixed point of $f_\lambda(z)$. Further, in this section, it is found that, under a certain condition, $\mathcal{J}(f_\lambda)$ is a nowhere dense subset of the right half plane when $0 < \lambda < \lambda^*$. In Section 4.5, the dynamical behaviour of $f_\lambda(z)$ for $\lambda > \lambda^*$ is described. We prove that the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$ contains the entire real line. The characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of escaping points, analogous to the first characterization in Section 4.4 is obtained in this case also. In Section 4.6, the characterizations of the Julia set of $f_\lambda(z)$, obtained in Sections 4.4 and 4.5, are applied to computationally generate the pictures of the

Julia set of $f_\lambda(z)$ for different values of λ. Finally, the results obtained herein for the entire function $f_\lambda(z) = \lambda\ (e^z - 1)/z$ are compared with those of Devaney [26, 31], Devaney and Durkin [33], Devaney and Krych [36], Devaney and Tangerman [37] and Misiurewicz [75] obtained for the dynamics of the critically finite entire functions $E_\lambda(z) = \lambda\ e^z$ of order one. The results of this chapter, some of them in weaker form, have appeared in [63].

4.2 Basic properties of functions $f_\lambda \in \mathcal{K}$

In this section some of the basic properties of the function $f_\lambda(z) = \lambda(e^z - 1)/z$, $\lambda > 0$ are developed. In Proposition 4.2.1, it is shown that $f_\lambda(z)$ maps the left half plane into an open disk centered at origin and having radius λ. The critical points play an important role in the dynamics of a function. In Proposition 4.2.2, it is proved that all the critical points of $f_\lambda(z)$ (*i.e.*, the points where $f'_\lambda(z) = 0$) are contained in the left half plane. In Proposition 4.2.3, the function $f_\lambda(z)$ is found to possess infinitely many critical points and critical values. Proposition 4.2.4 shows that the function $f_\lambda(z)$ is one-to-one in any closed rectangle of the form $R_{a,b,c} = \{z = x + iy : 2 \leq a \leq x \leq b, c \leq y \leq c + 2\pi\}$. Further, in this section, Proposition 4.2.5 and Proposition 4.2.6 endeavour to find certain domains in the right half plane for which $f_\lambda(z)$ is a homeomorphism. In Proposition 4.2.7, it is proved that the preimages of real points are dense in the set of all escaping points.

We begin by proving a mapping property of $f_\lambda(z)$:

Proposition 4.2.1. *Let $f_\lambda \in \mathcal{K}$. Then, $f_\lambda(z)$ maps the left half plane $H^- = \{z \in \mathbb{C} : \Re(z) <$ into an open disk centered at origin and having radius λ.*

Proof. For an arbitrarily fixed $z \in H^-$, let $g(z) = e^z$ and γ be the line segment defined by $\gamma(t) = tz$, $t \in [0, 1]$. Then,

$$\int_\gamma g(z)dz = \int_0^1 g(\gamma(t))\gamma'(t)dt = e^z - 1\,.$$

Since $M \equiv \max_{t\in[0,1]} |g(\gamma(t))| = \max_{t\in[0,1]} \left|e^{tz}\right| < 1$, for $z \in H^-$,

$$|e^z - 1| = \left|\int_\gamma g(z)dz\right| \le M \,.\, (\text{length of } \gamma) < |z| \,.$$

Thus, $\left|\frac{e^z-1}{z}\right| < 1$ for all $z \in H^-$. Consequently, $|f_\lambda(z)| = \lambda \left|\frac{e^z-1}{z}\right| < \lambda$ for all $z \in H^-$. □

Proposition 4.2.2. *Let $f_\lambda \in \mathcal{K}$. Then, the function $f'_\lambda(z)$ has no zeros in the right half plane $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$.*

Proof. The function $f'_\lambda(z) = \lambda\frac{e^z(z-1)+1}{z^2} = 0$ for $z \in H^+$ if and only if $e^z\,(z-1)+1 = 0$ for $z \in H^+$ if and only if $e^{-z} + (z-1) = 0$ for $z \in H^+$.

Define, for all $z \in \mathbb{C}$,

$$\begin{aligned} g_n(z) &= e^{-z} + z - \left(1 + \frac{1}{n}\right), \quad n = 1, 2, 3, \cdots \\ g(z) &= e^{-z} + z - 1 \,. \end{aligned}$$

The sequence $\{g_n(z)\}$ converges to $g(z)$ uniformly on every compact subset of $\mathbb{C}$. We first show that for each $n \ge 1$, $g_n(z)$ has only one real zero in the region $\Omega_n(R) = \{z \in \mathbb{C} : |z| \le R,\ \Re(z) \ge 0,\ R > 2 + \frac{1}{n}\}$.

Let $\phi_n(z) = z - \left(1 + \frac{1}{n}\right)$ and $h(z) = e^{-z}$. For $R > 2 + (1/n)$, choose $\Gamma_n(R)$ to be the boundary curve of $\Omega_n(R)$, *i.e.*,

$$\begin{aligned} \Gamma_n(R) &= \{z \in \mathbb{C} : |z| = R,\ \Re(z) \ge 0\} \cup \{z \in \mathbb{C} : z = iy,\ -R \le y \le R\} \\ &= C_n(R) \cup D_n(R)\ (say)\,. \end{aligned}$$

For $z \in C_n(R)$,

$$|\phi_n(z)| \ge |z| - \left(1 + \frac{1}{n}\right) > 1 \ge |\, e^{-z}\,| = \left|\, e^{-\Re(z)}\,\right| = |h(z)| \,,$$

and, for $z \in D_n(R)$,

$$|\phi_n(iy)| = \left|iy - \left(1 + \frac{1}{n}\right)\right| = \sqrt{y^2 + \left(1 + \frac{1}{n}\right)^2} > 1 = |\, e^{-iy}\,| = |h(iy)| \,.$$

Thus, $|h(z)| < |\phi_n(z)|$ on $z \in \Gamma_n(R)$. Since $\phi_n(z)$ has only one zero in $\Omega_n(R)$, by Rouche's theorem, $g_n(z) = \phi_n(z) + h(z)$ also has only one zero w_n (say) in $\Omega_n(R)$. Since $g_n(0) < 0$ and $g_n(2 + (1/n)) > 0$, the zero w_n of $g_n(z)$ must be real. The sequence $\{w_n\}_{n=1}^{\infty}$ is clearly bounded. Therefore, there exists a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ such that $w_{n_k} \to w$ as $k \to \infty$, $w \in \Omega_n(R)$. It now follows by Hurwitz theorem applied to the sequence $\{g_{n_k}(z)\}_{k=1}^{\infty}$ that $w \in \Omega_n(R)$ is the only zero of $g(z)$. It is easily seen that the function $e^{-x} + x - 1$, $x \in$ R , attains the global minimum value 0 at the point $x = 0$. Therefore, it follows that $w = 0$. Consequently the only zero of $g(z)$ in $\Omega_n(R)$ is at the origin. Since $R > 2 + (1/n)$ is arbitrary, it follows that the only zero of $g(z)$ in $\overline{H^+} = \{z \in$ C $: \Re(z) \geq 0\}$ is at the origin. Thus, $f'_\lambda(z)$ has no zeros in H^+. □

Proposition 4.2.3. *Let $f_\lambda \in \mathcal{K}$. Then, the function $f_\lambda(z)$ possesses infinitely many critical values all lying in the open disk centered at origin and having radius λ.*

Proof. Let $f_\lambda(z) = \lambda\ \Psi(z)$ where $\Psi(z) = (e^z - 1)/z$ for $z \neq 0$ and $\Psi(0) = 1$. Then, for $z \neq 0$, $f'_\lambda(z) = \lambda\Psi'(z) = \lambda(e^z(z-1)+1)/z^2 = \lambda(e^z - \Psi(z))/z$ and $f'_\lambda(0) = \lambda/2$ so that the critical points of $f_\lambda(z)$ are non-zero roots of the equation $\Psi(z) = e^z$. If z^* is any critical point then the corresponding critical value is given by $f_\lambda(z^*) = \lambda\Psi(z^*) = \lambda \exp(z^*)$.

Now,

$$\begin{aligned} f'_\lambda(z) = 0 \quad &\Longleftrightarrow \quad e^z(z-1)+1 = 0 \text{ and } z = x+iy \neq 0 \\ &\Longleftrightarrow \quad e^x\ ((x-1)\cos y - y\sin y) + 1 = 0 \quad \text{and} \\ &\qquad\quad\ e^x\ ((x-1)\sin y + y\cos y) = 0 \\ &\Longleftrightarrow \quad \frac{y}{\sin y} - \exp(y\cot y - 1) = 0, \ \text{ and } x = 1 - y\cot y \end{aligned} \qquad (4.2.1)$$

Define,

$$g(y) = \frac{y}{\sin y} - e^{(y\cot y - 1)} \quad , \ y \in \mathrm{R} \setminus \{n\pi : n = 0, \pm1, \pm2, \dots\} \qquad (4.2.2)$$

If $y \in [(2n+(1/2))\pi, (2n+1)\pi)$, $n = 0, 1, 2, \ldots$, then $(y/\sin y) \geq y$ and $y\cot y - 1 \leq 0$. Therefore, the function $g(y)$ is positive in the interval $[(2n+(1/2))\pi, (2n+1)\pi)$. The function $g(y)$ is obviously negative in the interval $((2n+1)\pi, 2(n+1)\pi)$. Further, $g((2n+(1/4))\pi) < 0$ and $g((2n+(1/2))\pi) > 0$ for all integers $n \geq 1$. Since $g(y)$ is continuous in $[(2n+(1/4))\pi, (2n+(1/2))\pi)$, there exist points $y_n \in [(2n+(1/4))\pi, (2n+(1/2))\pi)$, for $n = 1, 2, \ldots$, such that $g(y_n) = 0$. In view of, $g(-y) = g(y)$, it follows that there exists a point $y_{-n} \in [-(2n+(1/2))\pi, -(2n+(1/4))\pi)$, $n = 1, 2, \ldots$, such that $y_{-n} = -y_n$ and $g(y_{-n}) = 0$. Thus, there exists a sequence $\{y_n\}_{n=-\infty, n\neq 0}^{n=\infty}$ such that $g(y_n) = 0$. Set $x_n = 1 - y_n \cot(y_n)$ for $n = \pm1, \pm2, \ldots$. In view of (4.2.1), it follows that $f'_\lambda(z_n) = 0$ so that the points $z_n = x_n + iy_n$, $n = \pm1, \pm2, \ldots$ are critical points for $f_\lambda(z)$.

Define the critical points set $\mathcal{P}$ and the critical values set $\mathcal{V}$ as follows:

$$\begin{aligned}
\mathcal{P} &= \{z_n : f'_\lambda(z_n) = 0,\ z_n = x_n + iy_n,\ n = \pm1, \pm2, \ldots\} \\
\mathcal{V} &= \{f_\lambda(z_n) = \lambda\exp(z_n) : \quad z_n \in \mathcal{P}\}
\end{aligned}$$

Let, if possible, $f_\lambda(z)$ has finitely many critical values. Then, $\mathcal{V}$ is a finite set. This implies that

(i) $\mathcal{X} = \{x_n : z_n = x_n + iy_n \in \mathcal{P}\}$ must be a finite set, since if $\mathcal{X}$ is an infinite set then $\mathcal{V}$ is also an infinite set.

(ii) There exist positive integers M and N, a subsequence $\{m_k\}_{k=0}^{\infty}$ of non-negative integers with $m_0 = 0$ such that $m_k \to \infty$ as $k \to \infty$ and a subsequence $\{y_{n_k}\}_{k=0}^{\infty}$ of $\{y_n\}_{n=-\infty, n\neq 0}^{n=\infty}$ such that

$$\begin{aligned}
y_{n_k} &= 2m_k\pi + y_N \\
x_{n_k} &= x_M
\end{aligned}$$

Let $z_{n_k} = x_M + iy_{n_k} = x_M + i(y_N + 2m_k\pi)$. Then, since $f'_\lambda(z_{n_k}) = 0$, by (4.2.1) and

(4.2.2).

$$g(y_{n_k}) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{4.2.3}$$

Now, by (4.2.2), for any positive integer s,

$$g(y_N + 2s\pi) = \frac{y_N + 2s\pi}{\sin(y_N + 2s\pi)} - \exp\left\{(y_N + 2s\pi)\cot\left(y_N + 2s\pi\right) - 1\right\}.$$

Since $g(y_N) = g(y_{n_0}) = 0$,

$$\begin{aligned} g(y_N + 2s\pi) &= \frac{2s\pi}{\sin(y_N)} - [\exp(y_N \cot(y_N) - 1)]\,[\exp(2s\pi\cot(y_N)) - 1] \\ &< \frac{2s\pi}{\sin(y_N)} - [\exp\left(2s\pi\cot(y_N)\right) - 1]\,. \end{aligned} \tag{4.2.4}$$

But, (4.2.4) gives that for all sufficiently large values s, $g(y_N + 2s\pi) < 0$, which is a contradiction to (4.2.3). Thus $f_\lambda(z)$ has infinitely many critical values. The assertion that all critical values lie in the open disk centered at origin and having radius λ follows easily by Propositions 4.2.1 and 4.2.2. □

Remark 4.2.1. *(i) From Proposition 4.2.2, it follows that $f_\lambda(z)$ is locally one-to-one in the right half plane $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$.*

(ii) The function $f_\lambda(z)$ is not one-to-one in H^+, since two distinct points z_1 and z_2 in H^+ with $f_\lambda(z_1) = f_\lambda(z_2)$ may be constructed as follows: for a fixed $x > 0$, let l_k : $l_k(t) = (1-t)A_k + tB_k; 0 \le t \le 1$ be the line segment joining the points $A_k = x + i(2k\pi)$ and $B_k = x + i((2k+1)\pi)$, where k is any positive integer. Since $\Im(f_\lambda(A_k)) < 0$ and $\Im(f_\lambda(B_k)) > 0$, due to continuity of $\Im(f_\lambda(z))$, there exists a point z_k (say) on l_k such that $\Im(f_\lambda(z_k)) = 0$. Thus, $f_\lambda(z_k) = r_k$ (say) is a real number. In view of $f_\lambda(\overline{z}) = \overline{f_\lambda(z)}$, it follows that $f_\lambda(z_k) = f_\lambda(\overline{z_k}) = r_k$ where z_k and $\overline{z_k}$ are in H^+.

The following proposition shows that $f_\lambda(z)$ is one-to-one in any closed rectangle of the form $R_{a,b,c} = \{z = x + iy : a \le x \le b, c \le y \le c + 2\pi\}$ contained in $H_2 = \{z : \Re(z) \ge 2\}$. In particular, $f_\lambda(z)$ is one-to-one in any closed disk $B_\pi(z_0) \subseteq H_2$, having center at z_0 and radius π.

Proposition 4.2.4. *Let $f_\lambda \in \mathcal{K}$ and $H_2 = \{z \in \mathbb{C} : \Re(z) \geq 2\}$.*

(a) *For any vertical line segment Γ_1, contained in H_2 and having length 2π, $f_\lambda(\Gamma_1)$ is a starlike curve with respect to the origin (i.e., with parametric equation $\Gamma_1 : z(t)$, $0 \leq t \leq 1, \arg(f_\lambda(z(t)))$ is a non-decreasing function of t, for $t \in [0,1]$).*

(b) *For any horizontal line segment $\Gamma_2 = \{x + iy_0 : a \leq x \leq b, a, b \in \mathbb{R}$ and fixed $y_0 \in \mathbb{R}\}$ contained in H_2, $|f_\lambda(x + iy_0)|$ is an increasing function of x.*

(c) *$f_\lambda(z)$ is one-to-one on any closed rectangle*

$$R_{a,b,c} = \{z = x + iy : 2 \leq a \leq x \leq b, \quad c \leq y \leq c + 2\pi\}$$

Proof. (a) Let Γ_1 be the vertical line segment in H_2, joining the points $x_0 + i\gamma_0$ and $x_0 + i(\gamma_0 + 2\pi)$. Then, the parametric equation of Γ_1 is given by

$$\Gamma_1 : z \equiv z(t) = x_0 + i(\gamma_0 + 2\pi t), \; t \in [0,1] \, .$$

It is known ([49], vol.1, p110) that the image of Γ_1 under $f_\lambda(z)$ is a starlike curve with respect to the origin if and only if $\Im\left\{\frac{f'_\lambda(z(t))}{f_\lambda(z(t))} z'(t)\right\} \geq 0$ for $t \in [0,1]$.

Since $z'(t) = 2\pi i$, $f_\lambda(\Gamma_1)$ is starlike with respect to origin, if and only if

$$\Re\left\{\frac{f'_\lambda(z(t))}{f_\lambda(z(t))}\right\} \geq 0 \text{ for } t \in [0,1] \tag{4.2.5}$$

Now, for any $z = x + iy \in H_2$,

$$\begin{aligned} \Re\left\{\frac{f'_\lambda(z)}{f_\lambda(z)}\right\} &= \Re\left(\frac{e^z}{e^z - 1}\right) - \Re\left(\frac{1}{z}\right) \\ &= \frac{(1 - e^{-x} \cos y)}{(1 - e^{-x} \cos y) + (e^{-2x} - e^{-x} \cos y)} - \frac{x}{x^2 + y^2} \, . \end{aligned}$$

Thus, in view of (4.2.5), $f_\lambda(\Gamma_1)$ is starlike with respect to origin if and only if for $x_0 \geq 2$ and $\gamma_0 \leq y \leq \gamma_0 + 2\pi$,

$$1 + \frac{e^{-2x_0} - e^{-x_0} \cos y}{1 - e^{-x_0} \cos y} \leq x_0 + \frac{y^2}{x_0} \, . \tag{4.2.6}$$

Now, for $x_0 \geq 2$, the inequalities $|1 - e^{-x_0} \cos y| \geq |1 - e^{-2}|$ and $|e^{-x_0}(e^{-x_0} - \cos y)| \leq e^{-2}(e^{-2} + 1)$ hold for any $y \in \mathbb{R}$ and so it follows that

$$\left| \frac{e^{-2x_0} - e^{-x_0} \cos y}{1 - e^{-x_0} \cos y} \right| \leq \frac{e^{-2}(e^{-2} + 1)}{1 - e^{-2}} < 0.5 < x_0 - 1 + \frac{y^2}{x_0} .$$

The above inequality and (4.2.6) prove (a).

(b) Let $\Gamma_2 = \{x + iy_0 : a \leq x \leq b, a, b \in \mathbb{R}$ and fixed $y_0 \in \mathbb{R}\}$ be any horizontal line segment contained in H_2.

Define, for $x + iy_0 \in \Gamma_2$,

$$A_{y_0}(x) = |f_\lambda(x + iy_0)|^2 = \frac{\lambda^2 (1 + e^{2x} - 2 e^x \cos y_0)}{x^2 + y_0^2}$$

Since,

$$A'_{y_0}(x) = \frac{\lambda^2 [(2 e^{2x} - 2 e^x \cos y_0)(x^2 + y_0^2) - 2x(1 + e^{2x} - 2 e^x \cos y_0)]}{(x^2 + y_0^2)^2},$$

it follows that for $x \in [a, b]$,

$$A'_{y_0}(x) > 0 \iff e^x(x^2 + y_0^2) \geq \left((x^2 + y_0^2)\cos y_0 + xe^{-x} + xe^x - 2x \cos y_0\right) \quad (4.2.7)$$

Since, for $x \geq 2$,

$$\begin{aligned}
\left|\left((x^2 + y_0^2)\cos y_0 + x e^{-x} + x e^x - 2x \cos y_0\right)\right| &\leq x^2 + y_0^2 + x e^{-x} + x e^x + 2x \\
&\leq y_0^2 + xe^x(xe^{-x} + 1 + e^{-2x} + 2e^{-x}) \\
&\leq y_0^2 + xe^x\left(\frac{2}{e^2} + 1 + e^{-4} + 2e^{-2}\right) \\
&\leq y_0^2 + 2x e^x \\
&\leq (y_0^2 + x^2) e^x
\end{aligned}$$

the inequality in (4.2.7) follows. Thus, $A_{y_0}(x)$ is an increasing function of x, for $x + iy_0 \in \Gamma_2$. Consequently, $|f_\lambda(x + iy_0)|$ is also an increasing function of x, for $x + iy_0 \in \Gamma_2$, completing the proof of (b).

(c) Let $R_{a,b,c} = \{z = x + iy : 2 \le a \le x \le b, c \le y \le c + 2\pi\}$ be the closed rectangle contained in H_2. Let $\Gamma_{1,x}$ be the vertical line segment in $R_{a,b,c}$ joining the points $x + ic$ and $x + i(c + 2\pi)$.

By (a), $f_\lambda(\Gamma_{1,a})$ and $f_\lambda(\Gamma_{1,b})$ are starlike curves with respect to the origin. Further, it is easily seen that

$$\phi(x, c, \lambda) \le |f_\lambda(\Gamma_{1,x})| \le \psi(x, c, \lambda) \tag{4.2.8}$$

where

$$\phi(x, c, \lambda) = \left(\frac{\lambda^2 (e^{2x} + 1 - 2e^x)}{(c + 2\pi)^2 + x^2}\right)^{1/2} \text{ and } \psi(x, c, \lambda) = \left(\frac{\lambda^2 (e^{2x} + 1 + 2e^x)}{c^2 + x^2}\right)^{1/2}.$$

Let $a \ge 2$ be arbitrarily chosen. Since $\phi(x, c, \lambda) \to \infty$ as $x \to \infty$, there exists a number $b_0 \equiv b_0(a) > a$ such that $\psi(a, c, \lambda) < \phi(b, c, \lambda)$ for $b \ge b_0$. It now follows from (4.2.8) that $f_\lambda(\Gamma_{1,a})$ and $f_\lambda(\Gamma_{1,b})$ do not intersect each other for $b \ge b_0$. Thus, $f_\lambda(z)$ is one-to-one on the vertical line segments $\Gamma_{1,a}$ and $\Gamma_{1,b}$ for $b \ge b_0$ (See Figure 4.1).

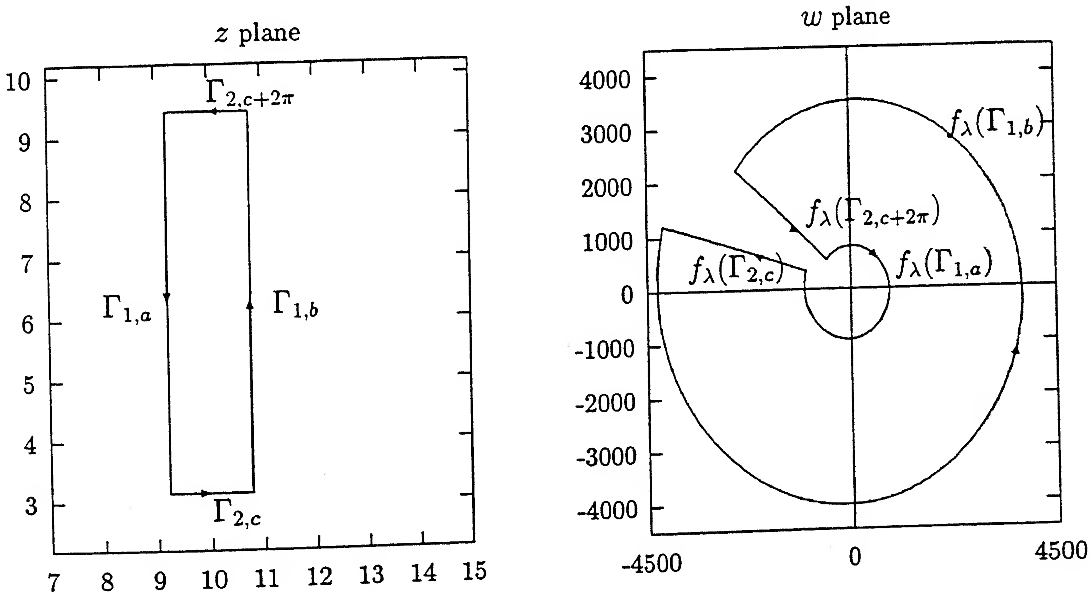


Figure 4.1: Image of the rectangle $R_{a,b,c}$, under the mapping $w = f_\lambda(z)$.

Let $\Gamma_{2,y}$ be the horizontal line segment in $R_{a,b,c}$ joining the points $a+iy$ and $b+iy$. We show that $f_\lambda(z)$ is one-to-one also on the horizontal boundary line segments $\Gamma_{2,c}$ and $\Gamma_{2,c+2\pi}$ of the rectangle $R_{a,b,c}$. Let $z_0 = x_0 + i(c+2\pi)$ be any arbitrarily fixed point on $\Gamma_{2,c+2\pi}$ and $z = x + ic$ be any point on $\Gamma_{2,c}$. Then,

$$\begin{aligned}
|f_\lambda(z) - f_\lambda(z_0)| = \lambda\left|\frac{e^z-1}{z} - \frac{e^{z_0}-1}{z_0}\right| &\geq \lambda\left|\frac{|e^z-1|}{|z|} - \frac{|e^{z_0}-1|}{|z_0|}\right| \\
&\geq \lambda\left|\frac{|z_0|\,|e^z-1| - |z|\,|e^{z_0}-1|}{|z_0|\,|z|}\right| \\
&\geq K(x,x_0)\left|\frac{|e^z-1| - |e^{z_0}-1|}{|z_0|\,|z|}\right| \qquad (4.2.9)
\end{aligned}$$

where, $K(x,x_0) = \lambda \min\{|z|, |z_0|\} > 0$. Now, if $a \leq x < x_0$, $|e^z-1| < |e^{z_0}-1|$ and if $x_0 < x \leq b$, $|e^{z_0}-1| < |e^z-1|$. Thus $||e^{z_0}-1| - |e^z-1|| > 0$ if $\Re(z) \in [a, x_0) \cup (x_0, b]$. Further, $|f_\lambda(z) - f_\lambda(z_0)| > 0$ for $\Re(z) = x_0 = \Re(z_0)$. Consequently, by (4.2.9), $|f_\lambda(z) - f_\lambda(z_0)| > 0$ for any $z \in \Gamma_{2,c}$. Since $z_0 \in \Gamma_{2,c+2\pi}$ is arbitrary, it follows that $f_\lambda(\Gamma_{2,c})$ and $f_\lambda(\Gamma_{2,c+2\pi})$ do not intersect. Further, by Proposition 4.2.4(b), $|f_\lambda(x+ic)|$ and $|f_\lambda(x+i(c+2\pi))|$ are increasing functions of x, for $x \geq 2$. Thus, $f_\lambda(z)$ is one-to-one on $\Gamma_{2,c} \bigcup \Gamma_{2,c+2\pi}$ (See Figure 4.1).

Let $\Delta_{\Gamma_{1,\mu}} \arg(f_\lambda(z_\mu(t))) = \arg(f_\lambda(z_\mu(1)) - \arg(f_\lambda(z_\mu(0)), \mu = a, b$, be the change in the argument of $f_\lambda(\Gamma_{1,\mu})$ as $z_\mu(t)$ traverses on $\Gamma_{1,\mu}$ from $(\mu + ic)$ to $(\mu + i(c+2\pi))$, where $z_\mu(t)$, $0 \leq t \leq 1$, is the parametric equation of $\Gamma_{1,\mu}$. For any non-zero $z \in \mathbb{C}$, $\arg(f_\lambda(z)) = \arg(e^z - 1) - \arg(z)$. Since $\Delta_{\Gamma_{1,\mu}} \arg(e^{z_\mu(t)} - 1) = 2\pi$ and $0 < \Delta_{\Gamma_{1,\mu}} \arg(z_\mu(t)) < \pi$, $\Delta_{\Gamma_{1,\mu}} \arg(f_\lambda(z_\mu(t)))$ satisfies the inequality $\pi \leq \Delta_{\Gamma_{1,\mu}} \arg(f_\lambda(z_\mu(t))) < 2\pi$ for $\mu = a, b$. Further, by Proposition 4.2.4(b), $|f_\lambda(x+iy)|$ for $y = c$ and $c+2\pi$, is an increasing function for $x \in [a,b]$ and, by Proposition 4.2.4(a), $f_\lambda(\Gamma_{1,\mu})$, $\mu = a, b$, is a starlike curve with respect to origin. Thus, it follows that $f_\lambda(z)$ is one-to-one on $\partial R_{a,b,c} = \Gamma_{1,a} \cup \Gamma_{1,b} \cup \Gamma_{2,c} \cup \Gamma_{2,c+2\pi}$, for $b \geq b_0$ (See Figure 4.1). Consequently ([74], vol.2, p118), $f_\lambda(z)$ is one-to-one in the closed rectangle $R_{a,b,c}$ for $b \geq b_0$. Since $R_{a,b,c} \subset R_{a,b_0,c}$ for $a \leq b < b_0$, $f_\lambda(z)$ is one-to-one on any closed rectangle $R_{a,b,c}$. This proves (c). □

Let $D_\delta(z_0)$ denote the open ball of radius δ, centered at z_0. If $\phi(z)$ is one-to-one, and $|\phi'(z)| > \mu > 1$, for all $z \in D_\delta(z_0)$, the function $\phi(z)$ expands the circular neighborhood $D_\delta(z_0)$ with scaling ratio greater than μ. The following proposition, the characteristic property of a locally one-to-one entire function $\phi(z)$ satisfying $|\phi'(z)| > \mu > 1$ to expand certain neighborhoods $U \subseteq D_\delta(z_0)$ of the point z_0 in such a way that $\phi(z)$ is a homeomorphism from U to $D_{\mu\delta}(\phi(z_0))$.

Proposition 4.2.5. *Let $\phi(z)$ be an entire function such that ϕ is one-to-one and $|\phi'(z)| > \mu > 1$ for all $z \in D_\delta(z_0)$, then there exists an open set $U \subseteq D_\delta(z_0)$ such that $\phi : U \to D_{\mu\delta}(\phi(z_0))$ is a homeomorphism.*

Proof. Since $\phi : D_\delta(z_0) \to \phi(D_\delta(z_0))$ is a bijection, its inverse map $L : \phi(D_\delta(z_0)) \to D_\delta(z_0)$ exists. Since $\phi(z)$ is analytic in $D_\delta(z_0)$, $L(z)$ is analytic in $\phi(D_\delta(z_0))$ ([74], vol.2. p86). It therefore follows that $\phi : D_\delta(z_0) \to \phi(D_\delta(z_0))$ is a homeomorphism. Thus ([30], p326), $\phi(D_\delta(z_0))$ contains a disk of radius $\mu\delta$ centered at $\phi(z_0)$. Since $\phi : D_\delta(z_0) \to \phi(D_\delta(z_0))$ is continuous and one-to-one, there exists an open set $U \equiv \phi^{-1}(D_{\mu\delta}(\phi(z_0))) \subseteq D_\delta(z_0)$ such that $\phi : U \to D_{\mu\delta}(\phi(z_0))$ is a homeomorphism. □

Corollary 4.2.1. *Let $f_\lambda \in \mathcal{K}$ and $|f_\lambda'(z)| > \mu > 1$ for all $z \in D_\delta(z_0) \subseteq H_2$, where $\delta \leq \pi$ and $H_2 = \{z \in \mathbb{C} : \Re(z) \geq 2\}$. Then, there exists an open set $U \subseteq D_\delta(z_0)$ such that $f_\lambda : U \to D_{\mu\delta}(f_\lambda(z_0))$ is a homeomorphism.*

Proof. Since $D_\delta(z_0) \subseteq H_2$ and $\delta \leq \pi$, with suitable choices of a, b and c, there exists a rectangle $R_{a,b,c}$ containing the disk $D_\delta(z_0)$. Therefore, by Proposition 4.2.4(c), $f_\lambda(z)$ is one-to-one in $D_\delta(z_0) \subseteq H_2$. The corollary now follows immediately from Proposition 4.2.5. □

Proposition 4.2.6. *Let $f_\lambda \in \mathcal{K}$ and U be an open set containing z_0. Let $z_n = f_\lambda^n(z_0)$, $n = 1, 2, 3, \cdots$. Define $D = \left\{z \in \mathbb{C} : |f_\lambda'(z)| > \sqrt{2} \text{ and } \Re(z) \geq 2\right\}$. Suppose, the open disk $D_{\sqrt{2}\pi}(z_n) \subset D$ for $n = 0, 1, 2, \cdots$. Let $S_{2\pi}(z_n)$ be the interior of the square with center*

at z_n and having sides of length 2π, parallel to the coordinate axes. Then, there exists an integer $N > 0$ and open sets $U_n \subseteq U$ for $n > N$, such that $f_\lambda^n : U_n \to S_{2\pi}(z_n)$ is a homeomorphism.

Proof. Suppose $D_\delta(z_0) \subset U$. Choose N so that $\delta(\sqrt{2})^{N-1} < \pi$ and $\delta(\sqrt{2})^N \geq \pi$. A repeated application of Corollary 4.2.1 gives an open set $W \subseteq D_\delta(z_0)$ that is mapped homeomorphically onto $D_{\delta(\sqrt{2})^N}(f_\lambda^N(z_0)$ by f_λ^N. Clearly, $D_{\delta(\sqrt{2})^N}(z_N) \supseteq D_\pi(z_N)$.

Let $n = 1$. By Corollary 4.2.1, there exists an open set $V_1 \subseteq D_\pi(z_N)$ such that $f_\lambda : V_1 \to D_{\sqrt{2}\pi}(z_{N+1})$ is a homeomorphism. Similarly, for $n = 2$, an application of Corollary 4.2.1 twice, gives an open set $V_2 \subseteq D_{\pi/\sqrt{2}}(z_N)$ such that $f_\lambda^2 : V_2 \to D_{\sqrt{2}\pi}(z_{N+2})$ is a homeomorphism. Continuing this process, for each integer $n > 2$, an open set $V_n \subseteq D_{\pi/(\sqrt{2})^{n-1}}(z_N) \subset D_\pi(z_N)$ is obtained such that $f_\lambda^n : V_n \to D_{\sqrt{2}\pi}(z_{N+n})$ is a homeomorphism. Consequently, since $D_{\sqrt{2}\pi}(z_{N+n}) \supseteq S_{2\pi}(z_{N+n})$, there exists a smaller set $V_n' \subseteq V_n$ such that $f_\lambda^n : V_n' \to S_{2\pi}(z_{N+n})$ is a homeomorphism, for each integer $n > 0$. Now, set for each integer $n > 0$ $U_{N+n} \equiv f_\lambda^{-N}(V_n') \subset W \subset D_\delta(z_0)$. Since $f_\lambda^N : W \to D_{\delta(\sqrt{2})^N}(z_N)$ is a homeomorphism and $D_{\delta(\sqrt{2})^N}(z_N) \supseteq D_\pi(z_N) \supseteq V_n \supseteq V_n'$, it follows that $f_\lambda^N : U_{N-n} \to V_n'$ is a homeomorphism. Consequently, for each integer $n > 0$, $f_\lambda^{N+n} : U_{N+n} \subset U \to S_{2\pi}(z_{N+n})$ is a homeomorphism. Thus, for each integer $n > N$, there is an open set $U_n \subset U$ for which $f_\lambda^n : U_n \subset U \to S_{2\pi}(z_n)$ is a homeomorphism. □

Proposition 4.2.7. *Let $f_\lambda \in \mathcal{K}$ and $Esc(f_\lambda) =$ clo $\{z \in \mathbb{C} : f_\lambda^n(z) \to \infty\}$ be the closure of the set of escaping points of $f_\lambda(z)$. Suppose $z_0 \in Esc(f_\lambda)$ and U is any open set containing z_0. Then, there exist an integer $N > 0$ and points $z_1, z_2 \in U$ such that $f_\lambda^N(z_1)$ is a real number and $\Re(f_\lambda^N(z_2) < 0$.*

Proof. Let $z_0 \in Esc(f_\lambda)$ and U be any neighborhood of z_0. Then, either $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$ or there exists a point $\tilde{z} \in U$ such that $f_\lambda^n(\tilde{z}) \to \infty$ as $n \to \infty$. In the latter case, rename $\tilde{z}$ as z_0, so that without loss of generality, if $z_0 \in Esc(f_\lambda)$, it may be assumed

that $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$.

For any $z = x + iy \neq 0$, the absolute value of $f_\lambda(z)$ is given by

$$|f_\lambda(x+iy)| = \lambda \left(\frac{1 + e^{2x} - 2\, e^x \cos y}{x^2 + y^2} \right)^{1/2} .$$

It is easily seen that

$$|f_\lambda(x_0 + iy)| \to 0 \quad \text{as} \quad |y| \to \infty, \text{ for any fixed } x_0 \in \mathrm{R}$$

$$|f_\lambda(x + iy_0)| \to 0 \quad \text{as} \quad x \to -\infty, \text{ for any fixed } y_0 \in \mathrm{R}$$

$$|f_\lambda(x + iy_0)| \to \infty \quad \text{as} \quad x \to \infty, \text{ for any fixed } y_0 \in \mathrm{R} .$$

Consequently, if $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$ then both $\Re(f_\lambda^n(z_0)) \to \infty$ and $\Im(f_\lambda^n(z_0)) \to \infty$ as $n \to \infty$ can not hold simultaneously. Further, if $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$, then $\Re(f_\lambda^n(z_0)) \to \infty$ and $\Im(f_\lambda^n(z_0))$ remain bounded. Since $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$, there exists a positive integer N_0 such that $f_\lambda^{N_0}(z_0) = w_0$ and $D_{\sqrt{2}\pi}(f_\lambda^n(w_0)) \subseteq D = \{z \in \mathrm{C} : |f_\lambda'(z)| > \sqrt{2}$ and $\Re(z) \geq 2\}$ for all $n = 0, 1, 2, \cdots$. Let V be a neighborhood of w_0 such that $V \subseteq f_\lambda^{N_0}(U)$. Applying Proposition 4.2.6 to the point w_0 and V, an integer $n_0 > 0$ is found such that, if $n > n_0$, there exists an open set $V_n \subset V$ for which $f_\lambda^n : V_n \to S_{2\pi}(f_\lambda^n(w_0))$ is a homeomorphism.

Fix $n = n_1 > n_0$. By Proposition 4.2.4(a), any vertical line segment of length 2π in the open square $S_{2\pi}(f_\lambda^n(w_0))$ is mapped by $f_\lambda(z)$ to a curve, which is starlike with respect to origin. Further, it follows easily from the proof of Proposition 4.2.4(c) that the image curve of any vertical line segment of length 2π in $S_{2\pi}(f_\lambda^n(w_0))$ intersects both the real and the imaginary axis. Thus, $f_\lambda^{n_1+1}(V_n) \cap \mathrm{R} \neq \emptyset$. Therefore, there exists a point $\tilde{z}_1 \in V_n \subseteq V$ such that $f_\lambda^{n_1+1}(\tilde{z}_1)$ is a real number. By setting $z_1 = f_\lambda^{-N_0}(\tilde{z}_1)$, it follows that $f_\lambda^{N_0+n_1+1}(z_1)$ is a real number for $z_1 \in U$.

The square $S_{2\pi}(f_\lambda^n(w_0))$ meets one of the horizontal lines $L_k = \{z = x + iy : x \geq 2$ and $y = (2k+1)\pi\}$ for some integer k. Since $f_\lambda(L_k)$ is contained in $H^- = \{z \in \mathrm{C} : \Re(z) < 0\}$,

there exists a point $\tilde{z}_2 \in V_n \subseteq V$ such that $\Re(f_\lambda^{n_1+1}(\tilde{z}_2)) < 0$. By setting $z_2 = f_\lambda^{-N_0}(\tilde{z}_2)$, it follows that $\Re(f_\lambda^{N_0+n_1+1}(z_2)) < 0$, where $z_2 \in U$. $\square$

4.3 Bifurcation in the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$

The dynamics of $f_\lambda(x) = \lambda(e^x - 1)/x$ for $x \in \mathbb{R}$ and $\lambda > 0$, is investigated in this section. The nature of the fixed points of $f_\lambda(x)$ is described in Theorem 4.3.1. Theorem 4.3.2 describes the dynamics of $f_\lambda(x)$. It follows from Theorem 4.3.2 that there exists a parameter value $\lambda^* > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$, occurs at $\lambda = \lambda^*$.

Set, $\Psi(x) = (e^x - 1)/x$ for $x \in \mathbb{R} \setminus \{0\}$ and $\Psi(0) = 1$. It is easily seen that the functions $\Psi(x)$ and $\Psi'(x)$ are strictly increasing positive valued functions. Therefore, the function $\phi(x) = \Psi(x) - x\Psi'(x)$, is strictly decreasing in $[0, \infty)$ and is positive in $(-\infty, 0]$. Consequently, since $\Psi(0) = 1$ and $\Psi(2) < 2\Psi'(2)$, there exists a unique $x^* \in (0, 2)$ such that

$$\phi(x) \begin{cases} > 0 & \text{for } -\infty < x < x^* \\ = 0 & \text{for } x = x^* \\ < 0 & \text{for } x^* < x < \infty \end{cases} \tag{4.3.1}$$

Throughout in the sequel, we denote

$$\lambda^* = \frac{1}{\Psi'(x^*)} \tag{4.3.2}$$

where, x^* is the unique real root of the equation $\phi(x) = 0$.

The following theorem describes the nature of the fixed points of $f_\lambda(x)$ for $x \in \mathbb{R}$.

Theorem 4.3.1. *Let $f_\lambda(x) = \lambda(e^x - 1)/x$ for $x \in \mathbb{R}$ and $\lambda > 0$.*

(a) *If $0 < \lambda < \lambda^*$, $f_\lambda(x)$ has an attracting fixed point and a repelling fixed point.*

(b) *If $\lambda = \lambda^*$, $f_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^*$.*

(c) *If $\lambda > \lambda^*$, $f_\lambda(x)$ has no fixed points.*

Proof. The proof of this theorem is analogous to the proof of Theorem 3.3.1. and hence. is omitted. □

The following theorem describes the dynamics of $f_\lambda(x)$ for $x \in \mathsf{R}$.

Theorem 4.3.2. *Let $f_\lambda(x) = \lambda(\ e^x\ - 1)/x$ for $x \in \mathsf{R}$ and $\lambda > 0$.*

(a) *If $0 < \lambda < \lambda^*$, $f_\lambda^n(x) \to\ a_\lambda$ as $n \to \infty$ for $x <\ r_\lambda$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x >\ r_\lambda$, where a_λ and r_λ are the attracting and the repelling fixed points of $f_\lambda(x)$ respectively.*

(b) *If $\lambda =\ \lambda^*$, $f_\lambda^n(x) \to\ x^*$ as $n \to \infty$ for $x <\ x^*$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x >\ x^*$, where x^* , given by (4.3.2), is the rationally indifferent fixed point of $f_\lambda(x)$.*

(c) *If $\lambda > \lambda^*$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathsf{R}$.*

Proof. The proof is analogous to that of Theorem 3.3.2. and hence, is omitted. □

It follows from the above theorem that there exist a real parameter $\lambda^* > 0$, given by (4.3.2), such that bifurcation in the dynamics of $f_\lambda(x)$ for $x \in \mathsf{R}$ occurs at $\lambda = \lambda^*$.

Let

$$\hat{x_\lambda} = \begin{cases} r_\lambda & \text{if } 0 < \lambda < \lambda^* \\ x^* & \text{if } \lambda =\ \lambda^* \end{cases}$$

If $0\ <\ \lambda\ \leq\ \lambda^*$, it follows from Theorem 4.3.2 that under iteration of f_λ the orbits of all the points less than $\hat{x_\lambda}$ remain bounded and the orbits of all the points greater than $\hat{x_\lambda}$ become unbounded; while, if $\lambda > \lambda^*$, there is no real point whose orbit remain bounded under iteration of f_λ. Thus, bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathsf{R}$, occurs at the parameter value $\lambda =\ \lambda^*$. The numerical computation of the root x^* of the equation $\phi(x) \equiv \Psi(x) - x\Psi'(x) = 0$ by the bisection method gives $x^* \approx 1.594$. Thus, by (4.3.2) the critical parameter $\lambda^*\ \approx 0.64761$. The bifurcation diagram for the function $f_\lambda(x) = \lambda(e^x - x)/x$ is shown in Figure 4.4.

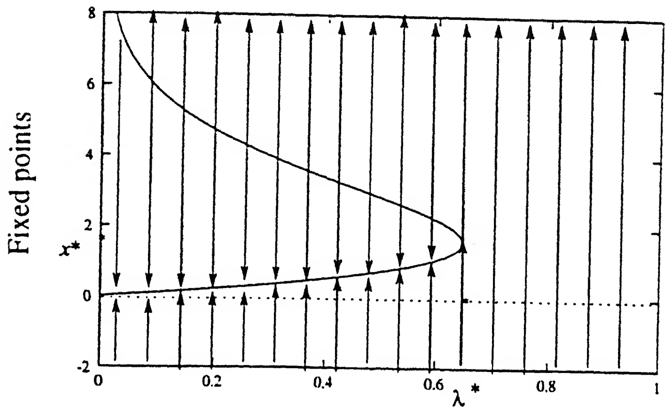


Figure 4.2: Bifurcation diagram for the function $f_\lambda(x) = \lambda(e^x - 1)/x$; $(\lambda > 0)$.

4.4 Dynamics of $f_\lambda(z)$ for $z \in \mathsf{C}$ and $0 < \lambda \le \lambda^* \approx 0.6476$

In this section, firstly, the dynamics $f_\lambda(z)$ for $z \in \mathsf{C}$ and $0 < \lambda < \lambda^*$ is described, where λ^* is defined by (4.3.2).

If $0 < \lambda < \lambda^*$, by Theorem 4.3.1(a) it follows that $f_\lambda(z)$ has a real attracting fixed point a_λ and a real repelling fixed point r_λ such that $\tilde{x}$ with $f'_\lambda(\tilde{x}) = 1$ satisfies $0 < a_\lambda < \tilde{x} < r_\lambda$. Let

$$\mathsf{A}(a_\lambda) = \{z \in \mathsf{C} : f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\} .$$

be the basin of attraction of the attracting fixed point a_λ of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$.

First, a general description of the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point a_λ of $f_\lambda(z)$, $0 < \lambda < \lambda^*$ is found in Proposition 4.4.1. Theorem 4.4.1 gives computationally useful characterization of the Julia set $\mathcal{J}(f_\lambda)$ as the closure of the set of escaping points of $f_\lambda(z)$. Theorem 4.4.2 provides another characterization of the Julia set $\mathcal{J}(f_\lambda)$ as the complement of the basin of attraction $\mathsf{A}(a_\lambda)$ of the attracting real fixed point of $f_\lambda(z)$. In Theorem 4.4.3, it is found under a certain condition, the Julia set $\mathcal{J}(f_\lambda)$, $0 < \lambda < \lambda^*$ is a nowhere dense subset of the right half plane.

We find in the following proposition that the basin of attraction $\mathsf{A}(a_\lambda)$ contains the

left half plane.

Proposition 4.4.1. *Let $f_\lambda \in \mathcal{K}$ and $0<\lambda<\lambda^*$. Then, the basin of attraction of the real attracting fixed point a_λ of $f_\lambda(z)$ contains the set $D = \{z \in \mathbb{C} : |f_\lambda(z)| < \tilde{x}\}$, where $\tilde{x}$ is the real number such that $f'_\lambda(\tilde{x}) = 1$. Further, $\tilde{x} > x^* \approx 1.594$, where x^* is given by (4.3.2).*

Proof. Let $f_\lambda(z) = \lambda\Psi(z)$ for $z \in \mathbb{C}$, where $\Psi(z) = (e^z - 1)/z$ and $\Psi(0) = \lambda$. Since $\frac{1}{\Psi'(\tilde{x})} = \lambda < \lambda^* = \frac{1}{\Psi'(x^*)}$ and $\Psi'(x)$ is strictly increasing function it follows that $\tilde{x} > x^* \approx 1.594$.

The function $f_\lambda(z)$ maps the open disk $D_{\tilde{x}}(0)$, centered at origin and having radius $\tilde{x}$ into itself can be shown analogous to the proof of Proposition 3.3.1. The function $f_\lambda(z)$ has zeros only at $2n\pi i$, for $n = \pm1, \pm2, \cdots$ and $|f_\lambda(z)| \leq \lambda < 1$ for $z \in \overline{H^-} = \{z \in \mathbb{C} : \Re(z) \leq 0\}$ and $0<\lambda<\lambda^*$. Since all the zeros of $f_\lambda(z)$ lie in $\overline{H^-}$, it follows that ([74], vol.1, p376) the curve $\gamma = \{z \in \mathbb{C} : |f_\lambda(z)| = \tilde{x}\}$ is connected and not self intersecting. Therefore $D = \{z \in \mathbb{C} : |f_\lambda(z)| < \tilde{x}\}$ is a simply connected domain. Since $f_\lambda(z)$ maps D into $D_{\tilde{x}}(0)$ and $f_\lambda(D_{\tilde{x}}(0)) \subseteq D_{\tilde{x}}(0)$, by Schwarz lemma ([30], p264), $f_\lambda^n(z) \to a_\lambda$ as $n \to \infty$ for all $z \in D$. Thus, $\mathsf{A}(a_\lambda) \supset D$. □

Remark 4.4.1. *(i) The left half plane $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$ is clearly contained in the basin of attraction $\mathsf{A}(a_\lambda)$, since by Proposition 4.2.1, $f_\lambda(H^-) \subset D_\lambda(0)$.*

(ii) The function $f_\lambda(z)$ has only one (finite) asymptotic value namely 0. The preimage of every neighborhood of a finite asymptotic value is an unbounded set. Since, the forward orbit of the asymptotic value 0 is attracted by the attracting fixed point a_λ of $f_\lambda(z)$, $0<\lambda<\lambda^*$, the basin of attraction $\mathsf{A}(a_\lambda)$ is unbounded. This contrasts the fact that every component of the Fatou set is bounded for the slow growth entire functions $A_\lambda(z)$ and $B_\lambda(z)$ considered in Chapter 3.

(iii) Consider horizontal strips S of the form $S = \{z : z = x+iy, x \in \mathbb{R}$ and

$|y| \in \left((2m+1)\pi, \left(\frac{4m+3}{2}\right)\pi\right), m = 0, 1, 2, \ldots\}$. *For* $z = x + iy \in S$ *and* $x \geq 0$, $\Re(f_\lambda(z)) = \frac{\lambda\, e^x}{x^2+y^2}\left[x \cos y + y \sin y - x\, e^{-x}\,\right] < 0$. *Therefore,* $\mathsf{A}(a_\lambda)$ *contains the horizontal strips* S. *Thus, the basin of attraction* $\mathsf{A}(a_\lambda)$ *contains* $D = \{z : |f_\lambda(z)| < \tilde{x}\} \cup S \supseteq H^- \cup S$. *Consequently,* $\mathsf{A}(a_\lambda)$ *occupies more than half of the complex plane and the complement of* $\mathsf{A}(a_\lambda)$ *is very small as compared to* $\mathsf{A}(a_\lambda)$ *(See white regions in Figures 4.4(a) and 4.5(a)).*

By Theorem 4.3.1(a), for $0 < \lambda < \lambda^*$, r_λ is the repelling fixed point for $f_\lambda(z)$ and therefore r_λ belongs to the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$. the Julia set $\mathcal{J}(f_\lambda)$. A characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of all escaping points of $f_\lambda(z)$ is found in Theorem 4.4.1.

Theorem 4.4.1. *Let* $f_\lambda \in \mathcal{K}$ *and* $Esc(f_\lambda) = clo\ \{z \in \mathsf{C} : f_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$ *be the closure of the set of escaping points of* $f_\lambda(z)$. *If* $0 < \lambda < \lambda^*$ *then the Julia set* $\mathcal{J}(f_\lambda) = Esc(f_\lambda)$.

Proof. Let $z_0 \in \mathcal{J}(f_\lambda)$ and U be any neighborhood of z_0. Since $\{f_\lambda^n\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \{f_\lambda^n(U)\}$ omits at most one point in C . In particular, there is a point $\hat{x} > r_\lambda$ such that $\hat{x} \in \bigcup_n \{f_\lambda^n(U)\}$, where r_λ is the repelling fixed point of $f_\lambda(z)$. Therefore, there exists a point $\hat{z} \in U$ such that $f_\lambda^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 4.3.2, it follows that $f_\lambda^n(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > r_\lambda$. Thus, there exists a point $\hat{z} \in U$ such that $f_\lambda^n(\hat{z}) \to \infty$ as $n \to \infty$ and $z_0 \in \mathrm{Esc}(f_\lambda)$ follows. Consequently, $\mathcal{J}(f_\lambda) \subseteq \mathrm{Esc}(f_\lambda)$.

To prove that $\mathrm{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$, we note that if $z_0 \in \mathrm{Esc}(f_\lambda)$ then $z_0 \notin \mathsf{A}(a_\lambda)$. Let U be any neighborhood of z_0. By Proposition 4.2.7, there exist an integer $N > 0$ and a point $\tilde{z} \in U$ such that $\Re(f_\lambda^N(\tilde{z})) < 0$, it therefore follows from Proposition 4.4.1 that $\tilde{z} \in \mathsf{A}(a_\lambda)$. Thus, in the neighborhood of z_0, there exists a point $\tilde{z} \in \mathsf{A}(a_\lambda)$. Now the orbit $\{f_\lambda^n(\tilde{z})\}$ of $\tilde{z}$ is bounded, while the orbit of z_0 escapes to ∞ under iteration of

f_λ . Thus, $\{f_\lambda^n\}$ is not normal at z_0. Consequently, $z_0 \in \mathcal{J}(f_\lambda)$ and $\text{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$ follows. □

Remark 4.4.2. *(i) The characterization of the Julia set of a function as the closure of the set of all escaping points of the function has so far been found only for certain critically finite entire transcendental functions [37]. Theorem 4.4.1 seems to be the first attempt to find such a characterization for non-critically finite entire transcendental functions $f_\lambda(z)$. The characterization of the Julia set found in Theorem 4.4.1 is quite useful in computationally generating the pictures of the Julia set of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$.*
(ii) Since, by Theorem 4.3.2(a), all the points $x > r_\lambda$ are escaping points of $f_\lambda(z)$, it follows that the interval (r_λ, ∞) is contained in the Julia set of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$.

The following Theorem 4.4.2 shows that $\mathcal{F}(f_\lambda)$ has only one component which is a basin of attraction of an attracting fixed point of $f_\lambda(z)$.

Theorem 4.4.2. *Let $f_\lambda \in \mathcal{K}$ and $0 < \lambda < \lambda^*$. Then, $\mathcal{F}(f_\lambda) = \mathsf{A}(a_\lambda)$, where $\mathsf{A}(a_\lambda)$ is the basin of attraction of the attracting real fixed point a_λ of $f_\lambda(z)$.*

Proof. If a point z_0 lies on an attracting cycle or a parabolic cycle of an entire transcendental function then the orbit of at least one of the singular values (*i.e.* critical values or asymptotic values) is attracted to the orbit of z_0 ([32], p182). Further, If U is a Siegel disk then the orbit of atleast one of the critical points is dense in the boundary of U ([32], p184). Now, since $f_\lambda(-x) \to 0$ as $x \to \infty$, it follows that 0 is the asymptotic value for $f_\lambda(z)$. By Proposition 4.2.3, all the singular values of $f_\lambda(z)$ lie in the open disk $D_\lambda(0)$ centered at origin and having radius λ. Since $f_\lambda(D_\lambda(0)) \subset f_\lambda(D_{\tilde{x}}(0)) \subseteq D_{\tilde{x}}(0)$, by Theorem 4.3.1 the orbits of all singular values lie in $D_{\tilde{x}}(0) \subset \mathsf{A}(a_\lambda)$, where $\mathsf{A}(a_\lambda)$ is the basin of attraction of the real attracting fixed point a_λ of $f_\lambda(z)$. Consequently, all singular values of $f_\lambda(z)$ and their orbits lie in the same component of $\mathsf{A}(a_\lambda)$. Therefore, it follows that $f_\lambda(z)$ has no parabolic domains and no Siegel disks in $\mathcal{F}(f_\lambda)$. Further, $\mathcal{F}(f_\lambda)$ has no attractive

basins other than $A(a_\lambda)$. By Theorem 4.4.1, it follows that $f_\lambda(z)$ do not have domain at infinity. If U is a wandering domain of $f_\lambda(z)$ then all the finite limit functions of $\{f^n|_U\}$ are contained in the derived set of forward orbits of all singular values of $f_\lambda(z)$ ([20], Theorem 1.1.10). Let U_0 be one of the component of wandering domain and $w \in U_0$. Since $\{f_\lambda^n(z)\}$ is normal in some neighborhood $N(w)$ of w. The limit function of $\{f_\lambda^n(z)|_{N(w)}\}$ can not be infinite. If the limit function of $\{f_\lambda^n(z)|_{N(w)}\}$ is finite, then the limit function is contained in the derived set of forward orbits of all singular values. But the forward orbits of all singular values tend to the attracting fixed point $A(a_\lambda)$. Therefore, the function $f_\lambda(z)$, $0 < \lambda < \lambda^*$, can not have a wandering domain. Thus, the only possible stable component U of $\mathcal{F}(f_\lambda)$ is the basin of attraction $A(a_\lambda)$ of the real attracting fixed point a_λ. □

Remark 4.4.3. *(i) It follows from Theorem 4.4.2 that* $\mathcal{J}(f_\lambda) = (A(a_\lambda))^c$, *giving another characterization of the Julia set as the complement of basin of attraction of non-critically finite entire transcendental function* $f_\lambda(z)$ *for* $0 < \lambda < \lambda^*$.
(ii) The basin of attraction $A(a_\lambda)$ *is a completely invariant set, since* $\mathcal{F}(f_\lambda)$ *is completely invariant and by Theorem* 4.4.2 $\mathcal{F}(f_\lambda) = A(a_\lambda)$.
(iii) Theorem 4.4.2, *under the condition* $D^* = \{z \in \mathbb{C} : |f'_\lambda(z)| < 1\} \subset A(a_\lambda)$, *has been proved earlier in [63].*

The following theorem shows that if $D^* = \{z \in \mathbb{C} : |f'_\lambda(z)| < 1\}$ then $A(a_\lambda)$ dense subset of $\mathbb{C}$.

Theorem 4.4.3. *Let* $f_\lambda \in \mathcal{K}$ *and* $D^* = \{z \in \mathbb{C} : |f'_\lambda(z)| < 1\}$ *be the proper subset of the basin of attraction* $A(a_\lambda)$ *of the real attracting fixed point* a_λ *of* $f_\lambda(z)$ *for* $0 < \lambda < \lambda^*$. *Then, the basin of attraction* $A(a_\lambda)$ *is a dense subset of* $\mathbb{C}$.

Proof. Let $z_0 \in (A(a_\lambda))^c$ and U be any open set containing z_0. We claim that U intersects the basin of attraction $A(a_\lambda)$. If $U \cap A(a_\lambda) = \emptyset$ then $f_\lambda^n(U) \cap A(a_\lambda) = \emptyset$ for all n.

Since $U \subset (A(a_\lambda))^c$, the inequality $|f'_\lambda(z)| > 1$ holds for all $z \in U$. Choose $\delta > 0$ so that the open disk $D_\delta(z_0) \subset U$ and $f_\lambda(z)$ is one-to-one in $D_\delta(z_0)$. Let μ_1 be such that $|f'_\lambda(z)| > \mu_1 > 1$ for all $z \in D_\delta(z_0)$. By Proposition 4.2.5 applied to $D_\delta(z_0)$, it follows that $f_\lambda(D_\delta(z_0)) \supset D_{\mu_1\delta}(f_\lambda(z_0))$. Now $D_{\mu_1\delta}(f_\lambda(z_0))$ does not meet $A(a_\lambda)$. Let μ_2 be such that $|f'_\lambda(z)| > \mu_2 > 1$ for all $z \in D_{\mu_1\delta}(f_\lambda(z_0))$. Again, by Proposition 4.2.5, we get $f_\lambda(D_{\mu_1\delta}(f_\lambda(z_0))) \supset D_{\mu_1\mu_2\delta}(f_\lambda^2(z_0))$. Continuing this process, and using the fact that $f_\lambda^n(U) \cap A(a_\lambda) = \emptyset$ for all n, it follows that there is an open disk $D_\rho(f_\lambda^n(z_0))$ of radius $\rho = \mu_1\mu_2\cdots\mu_n\delta$, center $f_\lambda^n(z_0)$ and contained in $f_\lambda^n(U)$, that does not meet $A(a_\lambda)$. If n is chosen large enough so that $\rho = \mu_1\mu_2\cdots\mu_n\delta \geq \pi$, then $D_\rho(f_\lambda^n(z_0))$ must meet one of the horizontal lines $L_k = \{z = x + iy : y = (2k+1)\pi, -\infty < x < \infty\}$ $k = 0, \pm1, \pm2, \cdots$, say L_{k_0}. But, the line L_{k_0} is mapped by $f_\lambda(z)$ to a curve in the left half plane $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$. This leads to a contradiction, since $A(a_\lambda) \supset H^-$. Therefore, $U \cap A(a_\lambda) \neq \emptyset$ and so $A(a_\lambda)$ is a dense subset of $\mathbb{C}$. □

Remark 4.4.4. *Since $A(a_\lambda)$ is a dense subset of $\mathbb{C}$ and contains the left half plane, the complement of $A(a_\lambda)$ is a nowhere dense subset of the right half plane. Thus, for $0 < \lambda < \lambda^*$, the Julia set of $f_\lambda(z)$ is a nowhere dense subset of the right half plane, if $D^* = \{z \in \mathbb{C} : |f'_\lambda(z)| < 1\} \subset A(a_\lambda)$.*

Remark 4.4.5. *The dynamics of $f_\lambda \in \mathcal{K}$ for $\lambda = \lambda^*$ is similar to that of the dynamics of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$ except for having parabolic domain corresponding to the rationally indifferent fixed point x^* in the Fatou set for $\lambda = \lambda^*$ instead of having basin of attraction as its Fatou set for $0 < \lambda < \lambda^*$.*

4.5 Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda^* \approx 0.6476$

In this section, we describe the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda^*$. In this case, we mainly prove a result analogous to Theorem 4.4.1 that characterizes the Julia

set of $f_\lambda(z)$ as the closure of the set of all escaping points. First we prove the following proposition giving that. for $\lambda > \lambda^*$, all the real points are contained in the Julia set $\mathcal{J}(f_\lambda)$.

Proposition 4.5.1. *Let $f_\lambda \in \mathcal{K}$ and $\lambda > \lambda^*$. Then, the real line $\mathbb{R}$ is contained in the Julia set $\mathcal{J}(f_\lambda)$ for $\lambda > \lambda^*$.*

Proof. Let $x_0 \in \mathbb{R}$ and U be any open set containing x_0. Set $x_n = f_\lambda^n(x_0)$, $n = 1, 2, 3, \cdots$. By Theorem 4.3.2(c), it follows that $x_n \to \infty$ as $n \to \infty$. Since $f'_\lambda(x) \neq 0$, for all $x \in \mathbb{R}$, $f_\lambda(x)$ is locally one-to-one for all $x \in \mathbb{R}$. Further, $f_\lambda(x) > x$ for $x \in \mathbb{R}$ and $f'_\lambda(x) \to \infty$ as $x \to \infty$. Therefore, using the continuity of $f'_\lambda(z)$, there exist an integer $N_0 > 0$ and an open ball $D_\delta(x_0) \subseteq U$ such that

(i) $f_\lambda^{N_0} : D_\delta(x_0) \to V$ is a homeomorphism.

(ii) $D_{\sqrt{2}\,\pi}(x_m) \subset \left\{z \in \mathbb{C} : |f'_\lambda(z)| > \sqrt{2} \text{ and } \Re(z) \geq 2\right\}$ for $m \geq N_0$.

Proposition 4.2.6, applied to the point x_{N_0} and the open set V containing x_{N_0}, gives that there exists an integer $N_1 > 0$ such that, if $n > N_1$, there exists an open set $V_n \subseteq V$ for which $f_\lambda^n : V_n \to S_{2\pi}(f_\lambda^n(x_{N_0}))$ is a homeomorphism, where $S_{2\pi}(f_\lambda^n(x_{N_0}))$ is the interior of the square with center at $f_\lambda^n(x_{N_0})$ and having sides of length 2π, parallel to the co-ordinate axes. Since $f'_\lambda(x_n) \to \infty$ as $n \to \infty$, by Proposition 4.2.5, it follows that f_λ expands $S_{2\pi}(f_\lambda^n(x_{N_0}))$ with arbitrarily large scaling ratio as $n \to \infty$. Therefore, large enough $N_2 > N_1$ can be chosen so that $f_\lambda(S_{2\pi}(f_\lambda^{n-1}(x_{N_0}))) \supset S_{2\pi}(f_\lambda^n(x_{N_0}))$ for $n > N_2$.

Let $\Gamma : ab\overset{\frown}{cd}ef$ denote the boundary of the square $S_{2\pi}(f_\lambda^n(x_{N_0}))$ and $f_\lambda(\Gamma)$ be denoted by (See Figure 4.3). Since the mappings $f_\lambda^{N_0} : D_\delta(x_0) \to V$, $f_\lambda^n : V_n \to S_{2\pi}(f_\lambda^n(x_{N_0}))$

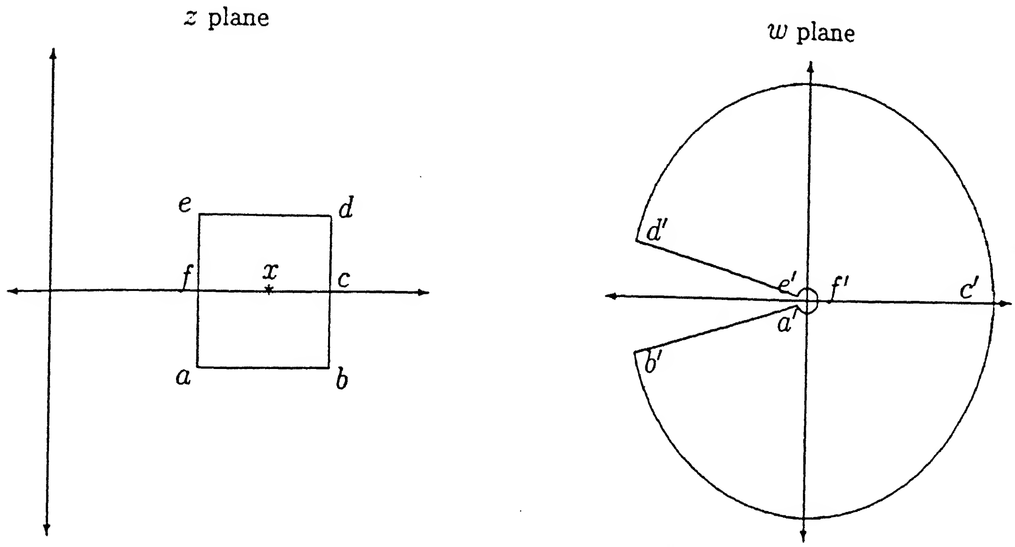


Figure 4.3: Image of the square $S_{2\pi}(x)$, under the mapping $w = f_\lambda(z)$.

$w_2 \in f_\lambda(S_{2\pi}(f_\lambda^n(x_{N_0})))$ sufficiently near the boundary point c', there exists a point in U that gets mapped to w_2 by $f_\lambda^{N_0+n+1}$ for each $n > N_1$. The points w_1 and w_2, in turn, get mapped by the function $f_\lambda(z)$ respectively to the points $f_\lambda(w_1)$ and $f_\lambda(w_2)$ one inside the open disk $D_\lambda(0)$ and the other arbitrarily close to ∞. Therefore, for each integer $n > N_0 + N_1 + 2$, there exist distinct points x_1 and x_2 in the open set U containing x_0 that get mapped by the function $f_\lambda^n(z)$ to the points $f_\lambda^n(x_1)$ and $f_\lambda^n(x_2)$ one inside the open disk $D_\lambda(0)$ and the other arbitrarily close to ∞ respectively. Thus, the family of functions $\{f_\lambda^n\}$ is not normal in U and so $x_0 \in \mathcal{J}(f_\lambda)$. Since x_0 is any arbitrary real point, it follows that $\mathcal{J}(f_\lambda)$ contains the real line R . □

Remark 4.5.1. *For $0 < \lambda < \lambda^*$, the Julia set $\mathcal{J}(f_\lambda)$ lies entirely in the right half plane $H^+ = \{z \in$ C $: \Re(z) > 0\}$. As soon as the real parameter λ crosses the value λ^* , there is a sudden change in the geometry of the Julia set of $f_\lambda(z)$ since, by the above proposition, the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$, spreads to the left half plane as well. Thus, bifurcation in the dynamics of $f_\lambda(z)$ for $\lambda > 0$ occurs at $\lambda = \lambda^*$ (≈ 0.64761).*

The following theorem, analogous to Theorem 4.4.1 provides a characterization for the

Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$:

Theorem 4.5.1. *Let $f_\lambda \in \mathcal{K}$ and $Esc(f_\lambda) = clo\ \{z \in \mathbb{C} : f_\lambda^n(z) \to \infty$ as $n \to \infty\}$ be the closure of the set of escaping points of $f_\lambda(z)$. If $\lambda > \lambda^*$ then, the Julia set $\mathcal{J}(f_\lambda) = Esc(f_\lambda)$.*

Proof. The inclusion relation $\mathcal{J}(f_\lambda) \subseteq \text{Esc}(f_\lambda)$ follows on the lines of proof similar to that of Theorem 4.4.1. We need only to prove $\text{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$.

Let $z_0 \in \text{Esc}(f_\lambda)$ and U be an open set containing z_0. By Proposition 4.2.7, it follows that there exists an integer $N > 0$ and a point $\hat{z} \in U$ such that $f_\lambda^N(\hat{z})$ is a real number. Therefore, by Proposition 4.4.1, $\hat{z} \in \mathcal{J}(f_\lambda)$. Thus, $\{f_\lambda^n\}$ is not normal in U. Thus, $\text{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$. This completes the proof of Theorem 4.4.1. □

4.6 Applications

The characterizations of the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$ in Theorems 4.4.1 and 4.5.1 give a useful algorithm to computationally generate the pictures of the Julia set of $f_\lambda(z)$. For $f_\lambda \in \mathcal{K}$, $\Re(f^n(z)) \to \infty$ as $n \to \infty$ implies that $|f^n(z)| \to \infty$ as $n \to \infty$. This is implemented in the algorithm. The algorithm is as follows:

1. Select a square in the plane and construct a $k \times k$ grid in this square.

2. For each grid point, compute the orbit of this point up to a maximum of N iterations.

3. If, at iteration $i < N$, the real part of the orbit is greater than some given bound M, the original grid point is colored black and the iterations are stopped.

4. If real part of no point in the orbit ever becomes greater than B, the original grid point is left as white.

Thus, in the output generated by this algorithm, the black points represent the Julia set of $f_\lambda(z)$ and the white points represent the Fatou set of $f_\lambda(z)$. Sometimes, either the real parts of the orbits of certain white points may take longer than N iterations to escape

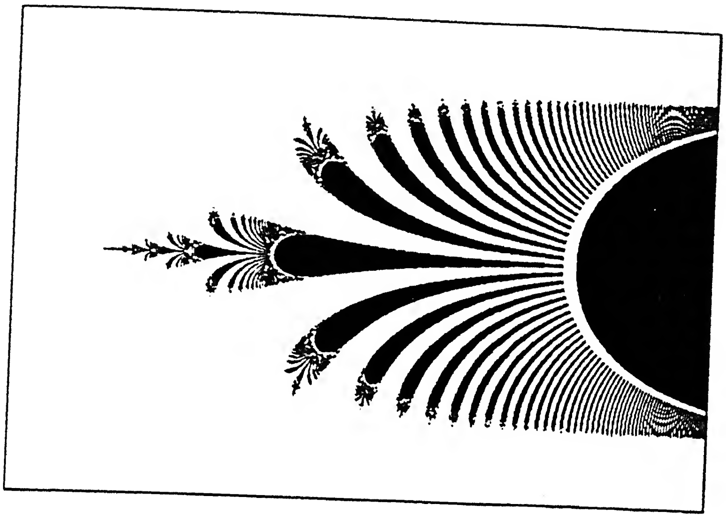
the bound M or the real parts of the orbits of certain black points may take longer than N iterations to become smaller than the given bound M. To overcome this difficulty, for $0 < \lambda < \lambda^*$, first a bound $M = M_0$ (say) is fixed and the pictures of the Julia set of $f_\lambda(z)$ with various values of N are computed. Then, by experimental observations, an integer $N_0 = N_0(M_0)$ is determined such that, for all $N > N_0$, the computed pictures of the Julia set of $f_\lambda(z)$ are similar for a maximum number of N iterations and $M = M_0$.

Using the above algorithm in the rectangular domain $R = \{z \in \mathbb{C} : 1.5 \leq \Re(z) \leq 8.5$ and $-2.5 \leq \Im(z) \leq 2.5\}$, the Julia set of $f_\lambda(z)$ for $\lambda = 0.64$ and $\lambda = 0.65$ are generated. To generate these pictures, for each grid point in the rectangle R the maximum number of iterations $N = 240$ is allowed for a possible escape of the bound $M = 100$. The generated pictures of the Julia sets for $\lambda = 0.64$ and $\lambda = 0.65$ are shown in Figure 4.4.

The Julia set of $f_\lambda(z)$ for $\lambda = 0.64 < \lambda^*$ is having the same pattern as those of the Julia sets of $f_\lambda(z)$ for all λ satisfying $0 < \lambda < \lambda^*$. Further, it is observed that the nature of picture of the Julia set $\mathcal{J}(f_{0.64})$ remains unaltered by increasing the maximum number of iterations $N \geq 200$ for a fixed bound $M = 100$. While the nature of picture of the Julia set of $f_\lambda(z)$ for $\lambda = 0.65 > \lambda^*$ shows a distinct change on increasing the number of iterations and, for a fixed bound $M = 100$, it becomes increasingly more black as the maximum number of iterations N is increased. Figure 4.4 suggests that the Julia set of $f_\lambda(z)$ admits Cantor bouquets for $0 < \lambda < \lambda^*$ and there is an explosion in the Julia set of $f_\lambda(z)$ as λ crosses the threshold value λ^*.

The following difference in the nature of the Julia set of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$ and the nature of the Julia set of $E_\lambda(z) = \lambda e^z$ for $0 < \lambda < \frac{1}{e}$, as found in [33], is observed. Since e^z is periodic, the Julia set of $E_\lambda(z)$ is the same in any horizontal strip of length 2π. The function $f_\lambda(z)$ is not periodic and $|f_\lambda(x_0 + iy)| \to 0$ as $|y| \to \infty$ for each fixed $x_0 \in \mathbb{R}$. Therefore, it follows that for each vertical line $L_x = \{z = x + iy : -\infty < y < \infty$ and fixed $x > 0\}$, there exists $y_x = y(x)$ such that $x + iy \in L_x$ satisfies $|f_\lambda(x + iy)| < \tilde{x}$ for $|y| < y_x$.

(a) $\lambda = 0.64 < \lambda^*$



(b) $\lambda = 0.65 > \lambda^*$

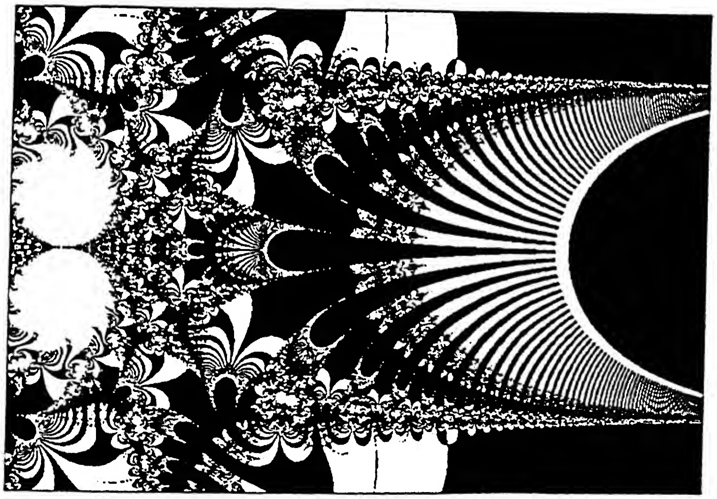
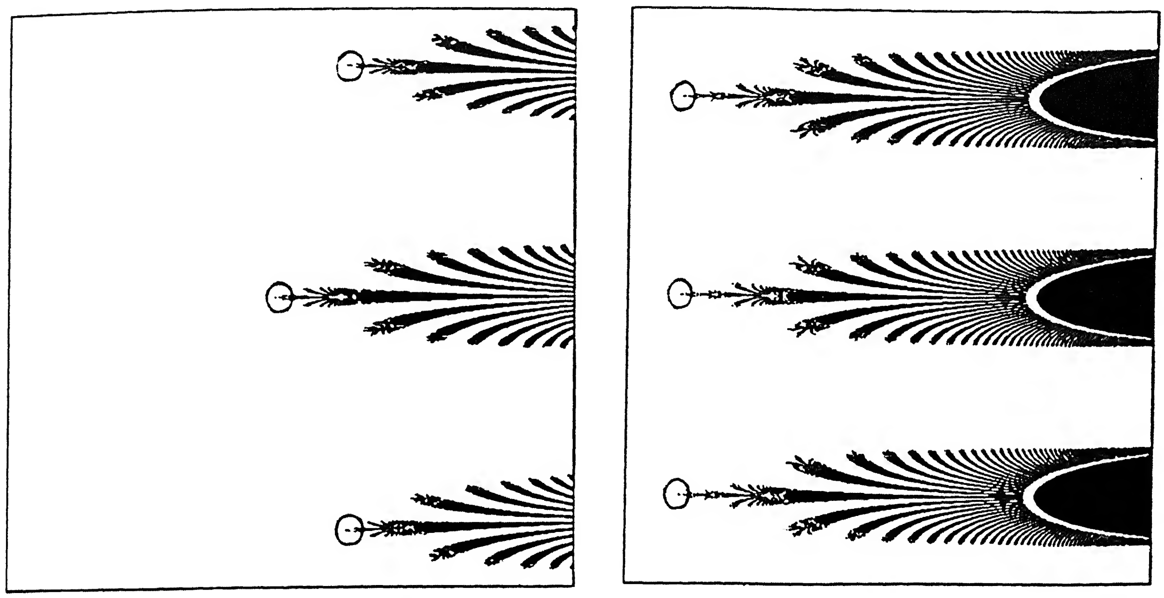


Figure 4.4: Explosion in the Julia set of $f_\lambda(z)$.



(a) Julia set of $f_{0.33}(z)$ (b) Julia set of $E_{0.33}(z)$

○ tip of the crown of the Cantor bouquet.

Figure 4.5: Comparison between the Julia sets of $f_{0.33}(z)$ and $E_{0.33}(z)$.

where $\tilde{x}$ is such that $f'_\lambda(\tilde{x}) = 1$. By Proposition 4.4.1, the points on L_x satisfying $|f_\lambda(x + iy)| < \tilde{x}$ lie in the Fatou set $\mathcal{F}(f_\lambda)$. Therefore, as the vertical distance from the real axis increases the real parts of the tip of the crowns of the Cantor bouquets of the Julia set of $f_\lambda(z)$, $0 < \lambda < \lambda^*$, are pushed more towards right, while as noted in the case of $E_\lambda(z)$, $0 < \lambda < \frac{1}{e}$, the real parts of the tip of the crowns of all the Cantor bouquets remain same irrespective of their distances from the real axis (See Figure 4.5).

Finally, a general comparison between the dynamical properties of non-critically finite function $f_\lambda(z)$ as found in the present chapter and critically finite function $E_\lambda(z) = \lambda e^z$ as found in ([26, 31, 33, 36, 37, 75]), is given in the following table:

$f_\lambda(z) = \lambda\frac{e^z-1}{z}$, $\lambda > 0$	$E_\lambda(z) = \lambda e^z$, $\lambda > 0$
$f_\lambda(z)$ is not periodic.	$E_\lambda(z)$ is periodic.
$f_\lambda(z)$ has infinitely many critical values.	$E_\lambda(z)$ has no critical values.
$f_\lambda(z)$ has only one asymptotic value, namely 0.	$E_\lambda(z)$ has only one asymptotic value, namely 0.
The bifurcation occurs in the dynamics of $f_\lambda(z)$ at the critical parameter value $\lambda^* \approx 0.64761$, defined by (4.3.2).	The bifurcation occurs in the dynamics of $E_\lambda(z)$ at the critical parameter value $\lambda^* = 1/e$.
For $0 < \lambda < \lambda^*$, $\mathcal{F}(f_\lambda)$ equals the basin of attraction of the real attracting fixed point.	For $0 < \lambda < \frac{1}{e}$, $\mathcal{F}(E_\lambda)$ equals the basin of attraction of the real attracting fixed point.
For $0 < \lambda < \lambda^*$, $\mathcal{J}(f_\lambda)$ lies only in the right half plane .	For $0 < \lambda < \frac{1}{e}$, $\mathcal{J}(E_\lambda)$ lies only in the right half plane .
$\mathcal{J}(f_\lambda)$ contains the real line R for $\lambda > \lambda^*$.	$\mathcal{J}(E_\lambda)$ equals the complex plane C for $\lambda > \frac{1}{e}$.
The real parts of the tip of the crowns of Cantor bouquets of $\mathcal{J}(f_\lambda)$, $0 < \lambda < \lambda^*$ are pushed towards right as their vertical distances from the real axis increases.	The tip of the crowns of Cantor bouquets of $\mathcal{J}(E_\lambda)$, $0 < \lambda < \frac{1}{e}$ have the same real part irrespective of their vertical distances from the real axis.
$\mathcal{J}(f_\lambda) = Esc(f_\lambda)$ for $\lambda > 0$.	$\mathcal{J}(E_\lambda) = Esc(E_\lambda)$ for $\lambda > 0$.

Table 4.1: Comparison between the dynamical properties $f_\lambda(z) = \frac{\lambda(e^z-1)}{z}$ and $E_\lambda(z) = \lambda e^z$.

Chapter 5

Dynamics of the non-critically finite even entire function $\sinh z/z$

The dynamics of even periodic entire function $\cos z$ which has only two critical values and no asymptotic (finite) value and hence is critically finite, is studied by Devaney and Durkin [33]. In this chapter, the dynamics of a non-critically finite even entire transcendental function is studied. For this purpose, the function $(\sinh z)/z$ is considered which is a non-critically even entire transcendental function arising as the denominator of a general T-fraction.

5.1 One parameter family $\mathcal{H} \equiv \{h_\lambda(z) = \frac{\lambda \sinh z}{z} : \lambda \in \mathbb{R} \setminus \{0\}\}$

For the general T-fraction,

$$D(z) \equiv \mathop{\mathrm{K}}_{n=1}^{\infty} \left(\frac{z^2/((2n)(2n+1))}{1-(z^2/((2n)(2n+1)))} \right), \tag{5.1.1}$$

let $B_n(z)$ denote the denominator of the nth approximant of the continued fraction (5.1.1). Since $\sum |\frac{1}{(2n)(2n+1)}| < \infty$, by Theorem 2.4.1, it follows that for all $z \in \mathbb{C}$ the sequence $\{B_n(z)\}_{n=1}^{\infty}$ converges to the even entire function $B(z) = 1 + \sum_{k=1}^{\infty} q_k z^{2k}$. It is readily seen

from (1.2.2) that the function $B_n(z)$ satisfies the recurrence relation

$$B_n(z) = B_{n-1}(z) + \frac{z^2}{(2n)(2n+1)}[B_{n-1}(z) - B_{n-2}(z)]$$

with initial conditions $B_{-1} \equiv 0$ and $B_0 \equiv 1$ and therefore, it is represented in the form $B_n(z) = 1 + \sum_{k=1}^{n} \frac{z^{2k}}{F_1F_2\cdots F_k}$ with $F_k = \frac{1}{(2k)(2k+1)}$, $k = 1, 2, \ldots$. Now,

$$\lim_{n\to\infty} B_n(z) = \lim_{n\to\infty}\left(1 + \sum_{k=1}^{n} \frac{z^{2k}}{F_1F_2\cdots F_k}\right) = 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{(2k+1)!} \equiv \frac{\sinh z}{z}.$$

Here and throughout in the sequel we assume that the function $\frac{\sinh z}{z}$ is defined to be equal to 1 for $z = 0$. Thus, the sequence $\{B_n(z)\}$ of denominators of the approximants of the modified general T-fraction (5.1.1) converges to the function $\sinh z/z$ for all $z \in \mathbb{C}$. We observe that, for the general T-fraction (5.1.1), giving rise to the even entire function $\frac{\sinh z}{z}$, $\sum |F_n| = \sum |G_n| = \sum \frac{1}{(2n)(2n+1)} < \infty$ while, as observed in Chapter 4, for the general T-fraction (4.1.1) that gave rise to the entire function $(e^z - 1)/z$, $\sum |F_n| = \sum |G_n| = \sum \frac{1}{n+1} = \infty$. Moreover, the entire functions considered in the present chapter are of order (c.f. Definition 1.3.1) one, while those considered in Chapter 3 are of order zero. Further, as in chapter 4, the sequence $\{A_n(z)\}$ of numerators of the approximants converges to the function $A(z) = 1 - B(z)$ for all $z \in \mathbb{C}$, since $A_n(z) = 1 - B_n(z)$. Therefore, the dynamics of the function $A(z)$ is derivable from the dynamics of the function $B(z)$ and vice versa. Thus, without loss of generality, we study the dynamics of the function $B(z)$.

Let

$$\mathcal{H} \equiv \left\{h_\lambda(z) = \lambda\frac{\sinh z}{z} : \lambda \in \mathbb{R} \setminus \{0\}\right\}$$

be one parameter family of entire transcendental functions. The present chapter is devoted to the study of the dynamical behaviour of the entire transcendental non-critically finite even functions $h_\lambda \in \mathcal{H}$. In Section 5.2, we develop some of the basic properties of the functions $h_\lambda \in \mathcal{H}$ needed in the subsequent sections. In particular, it is found in this section that $h_\lambda \in \mathcal{H}$ is non-critically finite. In Section 5.3, we describe the dynamics of

$h_\lambda(x)$, where $x \in$ R and $\lambda \neq 0$ is a real parameter. In this section, it is shown that there exists a critical parameter value $\lambda^{**} > 0$ such that bifurcation in the dynamics of $h_\lambda(x)$, $x \in$ R , occurs at $|\lambda| = \lambda^{**}(\approx 1.104)$. *i.e.*, if the parameter value crosses the value λ^{**} or $-\lambda^{**}$, then a dramatic change occurs in the dynamics of $h_\lambda(x)$. The dynamics of $h_\lambda \in \mathcal{H}$ for $z \in$ C and $0 < |\lambda| < \lambda^{**}$ is studied in Section 5.4. For this case, we prove two different characterizations for the Julia set of $h_\lambda(z)$. The first characterization gives the Julia set $\mathcal{J}(h_\lambda)$ for $0 < |\lambda| < \lambda^{**}$ as the closure of the set of escaping points; while the second characterization describes it as the complement of the basin of attraction of an attracting real fixed point of $h_\lambda(z)$. Further, in this section, it is found that, under a certain condition, the Julia set $\mathcal{J}(h_\lambda)$ is a nowhere dense subset of the complex plane when $0 < |\lambda| < \lambda^{**}$. In Section 5.5, the dynamical behaviour of $h_\lambda \in \mathcal{H}$ for $|\lambda| > \lambda^{**}$ is described. We prove that the Julia set of $h_\lambda(z)$ for $|\lambda| > \lambda^{**}$ contains the entire real line. The characterization of the Julia set of $h_\lambda(z)$ as the closure of the set of escaping points, analogous to the first characterization in Section 5.4, is obtained in this case also. In Section 5.6, the characterizations of the Julia set of $h_\lambda(z)$ obtained in Sections 5.4 and 5.5, are applied to computationally generate the pictures of the Julia set of $h_\lambda(z)$ for different values of λ. Finally, in this section, the results obtained in this chapter for the dynamics of $h_\lambda \in \mathcal{H}$ are compared with those of Devaney and Durkin [33], obtained for the dynamics of critically finite even entire function $C_\lambda(z) = \lambda i \cos z$, $\lambda \in$ R $\setminus \{0\}$; and further, a comparison between the results on the dynamics of $f_\lambda(z) = \lambda(e^z - 1)/z$ and $h_\lambda(z) = \lambda \sinh z/z$, as found in Chapter 4 and in the present chapter, is provided.

5.2 Basic properties of functions $h_\lambda \in \mathcal{H}$

In this section some of the basic properties of the function $h_\lambda(z) = \lambda \sinh(z)/z$, $\lambda \in$ R $\setminus \{0\}$ are developed. In Proposition 5.2.1, it is shown that $h_\lambda(z)$ is non-critically finite. Proposition 5.2.2 shows that the function $h_\lambda(z)$ is one-to-one in any closed rectangle of

the form $R_{a,b,c} = \{z = x + iy : a \leq x \leq b, c \leq y \leq c + 2\pi\}$ with either $a > 2$ or $b \leq -2$. Further, Proposition 5.2.3 and Proposition 5.2.4 in this section endeavour to find domains in the complex plane for which $h_\lambda(z)$ is a homeomorphism. In Proposition 5.2.5, it is proved that the preimages of real points and the preimages of purely imaginary points are dense in the set of all escaping points.

We begin by proving the non-critical finiteness of $h_\lambda(z)$:

Proposition 5.2.1. *Let $h_\lambda \in \mathcal{H}$. Then, $h_\lambda(z)$ possesses infinitely many critical values all lying in the closed disk centered at origin and having radius $|\lambda|$.*

Proof. Let $h_\lambda(z) = \lambda\,\Phi(z)$ where $\Phi(z) = \sinh(z)/z = \sin(iz)/(iz)$ for $z \neq 0$ and $\Phi(0) = 1$. Then, for $z \neq 0$, $h'_\lambda(z) = \lambda\Phi'(z) = \lambda\, i\, ((iz)\cos(iz) - \sin(iz))/(iz)^2 = \lambda\, i\, (\cos(iz) - \Phi(z))/(iz)$ so that the critical points of $h_\lambda(z)$ are roots of the equation $\cos(iz) - \Phi(z) = 0$. If z^* is any critical point then the corresponding critical value is given by $h_\lambda(z^*) = \lambda\,\Phi(z^*) = \lambda\,\cos(iz^*)$.

Now,

$$\begin{aligned} h'_\lambda(z) = 0 \quad &\Longleftrightarrow \quad iz\cos(iz) - \sin(iz) = 0 \text{ and } z = x + iy \neq 0 \\ &\Longleftrightarrow \quad iz = \tan(iz) \text{ and } z = x + iy \neq 0 \\ &\Longleftrightarrow \quad w = \tan w \text{ where } w = iz \neq 0 \qquad (5.2.1) \end{aligned}$$

We prove that $\tan z = z$ has only infinitely solutions in the complex plane $\mathbb{C}$. Let D_n be the square with vertices at $A = n\pi(-1+i)$, $B = n\pi(1+i)$, $C = n\pi(1-i)$ and $D = n\pi(-1-i)$, where n is a sufficiently large integer.

For $z = x + iy$,

$$\tan z = \frac{1}{i}\frac{(1 - e^{-2iz})}{(1 + e^{-2iz})} = \frac{1}{i}\frac{(1 - e^{-2ix}(e^{2y}))}{(1 + e^{-2ix}(e^{2y}))}\,.$$

on the horizontal side $AB : z(t) = n\pi(t+i)$, $-1 \leq t \leq 1$, of D_n, we have

$$|\tan z| = \left|\frac{1 - e^{-2n\pi ti}e^{-2n\pi}}{1 + e^{-2n\pi ti}e^{-2n\pi}}\right| \leq \frac{1 + e^{-2n\pi}}{1 - e^{-2n\pi}} \qquad (5.2.2)$$

When z lies on the vertical side $CB : z(t) = n\pi(1+it)$, $-1 \leq t \leq 1$, of D_n, we have

$$|\tan z| = \left|\frac{1 - e^{-2n\pi i}e^{-2n\pi t}}{1 + e^{-2n\pi i}e^{-2n\pi t}}\right| \leq \frac{1+e^{-2n\pi}}{1-e^{-2n\pi}} \tag{5.2.3}$$

Since $|\tan(-z)| = |\tan(z)|$, the inequalities (5.2.2) and (5.2.3) can be written for the sides CD and DA of the square D_n.

Therefore, for all $z \in D_n$,

$$|\tan z| \leq \frac{1+e^{-2n\pi}}{1-e^{-2n\pi}} .$$

If n is sufficiently large, the following inequality is hold:

$$|z| = |n\pi(\pm 1 \pm i)| = |n\pi\sqrt{2}| > \frac{1+e^{-2n\pi}}{1-e^{-2n\pi}} \tag{5.2.4}$$

Therefore, it follows that for a sufficiently large integer n,

$$|\tan z| < |z| \qquad \text{for } z \in D_n .$$

Therefore, by Rouche's theorem, the difference between the number of zeros and the number of poles of the function $(\tan z - z)$ inside the square D_n equals the difference between the number of zeros and the number of poles of the function z. Since $\tan z$ has $2n$ poles inside the square D_n, the equation $\tan z = z$ has precisely $2n+1$ zeros inside D_n. As $n \to \infty$, the equation $\tan z = z$ has infinitely many solutions in $\mathbb{C}$.

Now, we show that $\tan z = z$ has only real roots in $\mathbb{C}$. Equating real and imaginary parts of the equation $\tan z = z$, it is found that, for a nonzero $z = x + iy$,

$$\frac{\sin(2x)}{\cos(2x)+\cosh(2y)} = x \quad \text{and} \quad \frac{\sinh(2y)}{\cos(2x)+\cosh(2y)} = y .$$

This implies that

$$x \neq 0, \quad y \neq 0 \quad \text{and} \quad \frac{\sin(2x)}{x} = \frac{\sinh(2y)}{y} \tag{5.2.5}$$

or,

$$x \neq 0, \ \ y = 0 \qquad \text{or} \qquad x = 0, \ \ y \neq 0 \tag{5.2.6}$$

It is easy seen that, for $x \neq 0$ and $y \neq 0$ then $|\frac{\sin(2x)}{x}| < 2$ and $|\frac{\sinh(2y)}{y}| > 2$ so that (5.2.5) is not possible. Further, $x = 0$, $y \neq 0$ is not possible since $\tan(iy) = iy$ has no real solution. Thus, $\tan z = z$ implies $x \neq 0$, $y = 0$ so that $\tan z = z$ has only real solution in $\mathbb{C}$. Thus, the equation $\tan z = z$ has only infinitely many real roots in the complex plane.

In view of (5.2.1), it follows that $h'_\lambda(z)$ possesses infinitely many zeros in the complex plane and all are lying on the imaginary axis. Thus, $h_\lambda(z)$ possesses infinitely many critical points $\{iy_k\}_{k=-\infty}^{\infty}$. Since $h_\lambda(z)$ is an entire function, $y_k \to \infty$ as $|k| \to \infty$.

Since $h_\lambda(z)$ takes real values on the imaginary axis, all the critical values of $h_\lambda(z)$ are real. Obviously, $h'_\lambda(0) = 0$ and λ is the critical values corresponding to the critical point $z = 0$. Since $|h_\lambda(iy)| \leq |\lambda|$ for $y \in \mathbb{R}$, it follows that all the critical values of $h_\lambda(z)$ lie in the closed disk centered at origin and having radius $|\lambda|$.

It only remains to show that $h_\lambda(z)$ has infinitely many critical values. Now, since $h_\lambda(y_k) \neq 0$ for $k = 0, \pm 1, \pm 2, \ldots$ and $h_\lambda(iy_k) \to 0$ as $|k| \to \infty$, $h_\lambda(iy_k)$ are distinct for infinitely many values of k. Therefore, $h_\lambda(z)$ possesses infinitely many critical values. $\square$

From Proposition 5.2.1, it follows that $h_\lambda(z)$ is locally one-to-one in the right and left half planes $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$ and $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$. The following proposition shows that $h_\lambda(z)$ is one-to-one in any closed rectangle $R_{a,b,c} = \{z = x + iy : a \leq x \leq b, c \leq y \leq c + 2\pi\}$ contained in $H_2 = \{z : \Re(z) \geq 2\} \cup \{z : \Re(z) \leq -2\}$ and, in particular, $h_\lambda(z)$ is one-to-one in any closed disk $B_\pi(z_0)$ that is contained in H_2, and has centered at z_0 and radius π.

Proposition 5.2.2. *Let $h_\lambda \in \mathcal{H}$ and $H_2 = \{z : \Re(z) \geq 2\} \cup \{z : \Re(z) \leq -2\}$.*

(a) *For any vertical line segment Γ_1, contained in H_2 and having length 2π, $h_\lambda(\Gamma_1)$ is a starlike curve with respect to the origin (i.e., with parametric equation $\Gamma_1 : z(t)$,*

$0 \leq t \leq 1$, $\arg(h_\lambda(z(t)))$ *is a non-decreasing function of* t, *for* $t \in [0, 1]$*).*

(b) *For any horizontal line segment* $\Gamma_2 = \{x + iy_0 : a \leq x \leq b, a, b \in \mathrm{R}$ *and fixed* $y_0 \in \mathrm{R}\}$ *contained in* H_2, $|h_\lambda(x + iy_0)|$ *is an increasing function as* $|x|$ *increases.*

(c) $h_\lambda(z)$ *is one-to-one on any closed rectangle*

$$R_{a,b,c} = \{z = x + iy : a \leq x \leq b, \quad c \leq y \leq c + 2\pi\}$$

contained in H_2.

Proof. Let $H_2^+ = \{z : \Re(z) \geq 2\}$ and $H_2^- = \{z : \Re(z) \leq -2\}$.

(a) Let Γ_1 be the vertical line segment in H_2^+, joining the points $x_0 + i\gamma_0$ and $x_0 + i(\gamma_0 + 2\pi)$. Then, the parametric equation of Γ_1 is given by

$$\Gamma_1 : z \equiv z(t) = x_0 + i(\gamma_0 + 2\pi t), \; t \in [0, 1] \, .$$

As in Proposition 4.2.4, since $z'(t) = 2\pi i$, $h_\lambda(\Gamma_1)$ is starlike with respect to origin, if and only if

$$\Re\left\{\frac{h'_\lambda(z(t))}{h_\lambda(z(t))}\right\} \geq 0 \text{ for } t \in [0, 1] \tag{5.2.7}$$

Now, for any $z = x + iy \in H_2$,

$$\begin{aligned}\Re\left\{\frac{h'_\lambda(z)}{h_\lambda(z)}\right\} &= \Re\left(\frac{\cosh(z)}{\sinh(z)}\right) - \Re\left(\frac{1}{z}\right) \\ &= \frac{\cosh(x)\sinh(x)}{\sinh^2(x) + \sin^2(y)} - \frac{x}{x^2 + y^2} \, .\end{aligned}$$

Thus, in view of (5.2.7), $h_\lambda(\Gamma_1)$ is starlike with respect to origin if and only if for $x_0 \geq 2$ and $\gamma_0 \leq y \leq \gamma_0 + 2\pi$,

$$\tanh(x_0) + 2\sin^2(y)\text{cosech}(2x_0) \leq x_0 + \frac{y^2}{x_0} \, . \tag{5.2.8}$$

Now, for $x_0 \geq 2$, the inequalities $0 < |\tanh(x)| < 1$ and $|2\text{cosech}(2x)| \leq 1$ hold for any $y \in \mathrm{R}$ and so it follows that

$$\left|\tanh(x) + 2\sin^2(y)\text{cosech}(2x)\right| < 2 < x_0 + \frac{y^2}{x_0} \, .$$

This proves that $h_\lambda(\Gamma_1)$ is a starlike curve with respect to origin. If the vertical line segment Γ_1 is contained in H_2^- then $\Gamma_3 = -z(t)$, $t \in [0,1]$, where $z(t)$, $0 \le t \le 1$ is the parametric equation of Γ_1 is contained in H_2^+ and $h_\lambda(\Gamma_3)$ is a starlike curve with respect to origin. In view of $h_\lambda(z) = h_\lambda(-z)$, it follows that $h_\lambda(\Gamma_1)$ is a starlike curve with respect to origin. This proves (a).

(b) Let $\Gamma_2 = \{x + iy_0 : a \le x \le b, a, b \in \mathrm{R}$ and fixed $y_0 \in \mathrm{R}\}$ be any horizontal line segment contained in H_2^+.

Define, for $x + iy_0 \in \Gamma_2$,

$$A_{y_0}(x) = |h_\lambda(x+iy_0)|^2 = \frac{\lambda^2\left(\sin^2(y_0) + \sinh^2(x)\right)}{x^2 + y_0^2}$$

Since,

$$A'_{y_0}(x) = \frac{\lambda^2\left[\left(2\sinh(x)\cosh(x)\right)(x^2+y_0^2) - 2x\left(\sin^2(y_0) + \sinh^2(x)\right)\right]}{(x^2+y_0^2)^2},$$

it follows that for $x \in [a, b]$,

$$A'_{y_0}(x) > 0 \iff \sinh(2x)(x^2+y_0^2) \ \ge \ \left(2x\sin^2(y_0) + 2x\sinh^2(x)\right) \tag{5.2.9}$$

Since, for $x \ge 2$,

$$\begin{aligned}
\left|\left(2x\sin^2(y_0) + 2x\sinh^2(x)\right)\right| &\le \left|2x\sin^2(y_0)\right| + \left|2x\sinh^2(x)\right| \\
&\le 2xy_0 + 2x\sinh(x)\cosh(x) \\
&< \sinh(2x)y_0 + x\sinh(2x) \\
&\le \sinh(2x)y_0 + x^2\sinh(2x) \\
&\le (y_0^2 + x^2)\sinh(2x)
\end{aligned}$$

the inequality in (5.2.9) follows. Thus, $A_{y_0}(x)$ is an increasing function of x for $x+iy_0 \in \Gamma_2$. Consequently, $|\, h_\lambda\ (x+iy_0)|$ is also an increasing function of x, for $x + iy_0 \in \Gamma_2$. If the horizontal line segment $\Gamma_2 = \{x + iy_0 : a \le x \le b < 0, a, b \in \mathrm{R}$ and fixed $y_0 \in \mathrm{R}\}$ is

contained in H_2^- then $\Gamma_4 = \{-z : z \in \Gamma_2\}$ is contained in H_2^+ and $|\ h_\lambda\ (x+iy_0)|$ is an increasing function of x, for $x+iy_0 \in \Gamma_4$. In view of $h_\lambda(z)$ is an even function. it follows that $|\ h_\lambda\ (x+iy_0)|$ is a decreasing function of x, for $x+iy_0 \in \Gamma_2 \subseteq H_2^-$. It completes the proof of (b).

(c) Let $R_{a,b,c} = \{z = x+iy : 2 \leq a \leq x \leq b, c \leq y \leq c+2\pi\}$ be the closed rectangle contained in H_2^+. Let $\Gamma_{1,x}$ be the vertical line segment in $R_{a,b,c}$ joining the points $x+ic$ and $x+i(c+2\pi)$.

It follows from (a) that $h_\lambda(\Gamma_{1,a})$ and $h_\lambda(\Gamma_{1,b})$ are starlike curves with respect to the origin. It is easily seen that

$$\tilde{\phi}(x,c,\lambda) \leq |h_\lambda(\Gamma_{1,x})| \leq \tilde{\psi}(x,c,\lambda) \tag{5.2.10}$$

where

$$\tilde{\phi}(x,c,\lambda) = \left(\frac{\lambda^2 \sinh^2(x)}{(c+2\pi)^2+x^2}\right)^{1/2} \quad \text{and} \quad \tilde{\psi}(x,c,\lambda) = \left(\frac{\lambda^2\left(\cosh^2(x)+1\right)}{c^2+x^2}\right)^{1/2}.$$

Let $a \geq 2$ be arbitrarily chosen. Since $\tilde{\phi}(x,c,\lambda) \to \infty$ as $x \to \infty$, there exists a number $b_0 \equiv b_0(a) > a$ such that $\tilde{\psi}(a,c,\lambda) < \tilde{\phi}(b,c,\lambda)$ for $b \geq b_0$. It now follows from (5.2.10) that $h_\lambda(\Gamma_{1,a})$ and $h_\lambda(\Gamma_{1,b})$ do not intersect each other for $b \geq b_0$. Thus, $h_\lambda(z)$ is one-to-one on the vertical line segments $\Gamma_{1,a}$ and $\Gamma_{1,b}$ for $b \geq b_0$ (See Figure 5.1).

Let $\Gamma_{2,y}$ be the horizontal line segment in $R_{a,b,c}$ joining the points $a+iy$ and $b+iy$. We show that $h_\lambda(z)$ is one-to-one also on the horizontal boundary line segments $\Gamma_{2,c}$ and $\Gamma_{2,c+2\pi}$ of the rectangle $R_{a,b,c}$. Let $z_0 = x_0 + i(c+2\pi)$ be any arbitrarily fixed point on

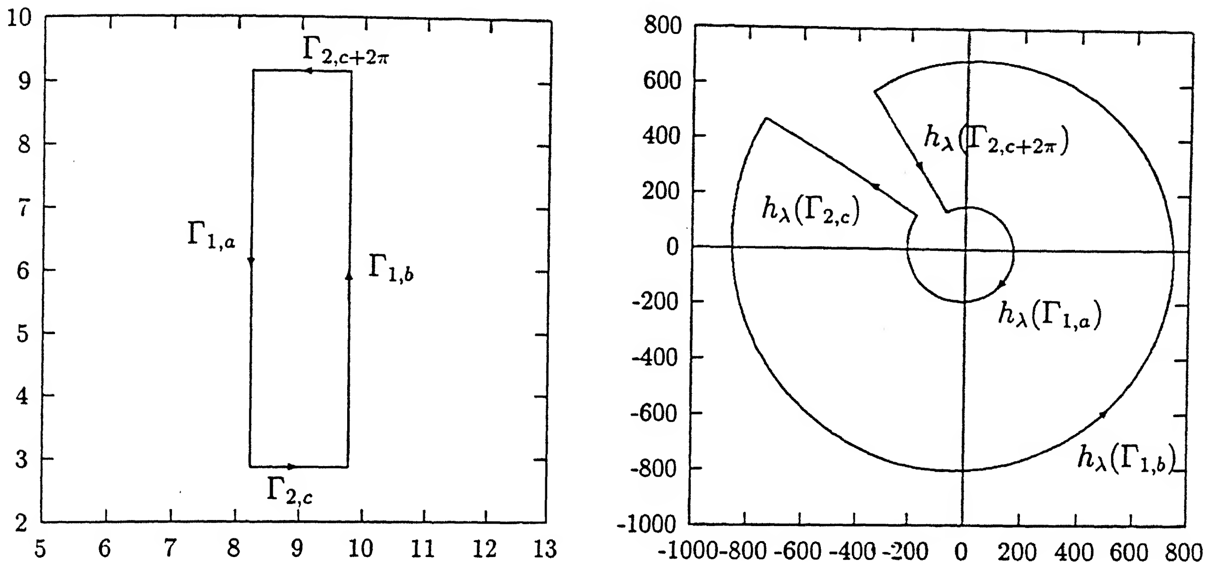


Figure 5.1: Image of the rectange $R_{a,b,c}$, under the mapping $w = h_\lambda(z)$.

$\Gamma_{2,c+2\pi}$ and $z = x + ic$ be any point on $\Gamma_{2,c}$. Then,

$$
\begin{aligned}
|h_\lambda(z) - h_\lambda(z_0)| = \lambda\left|\frac{\sinh z}{z} - \frac{\sinh z_0}{z_0}\right| &\geq \lambda\left|\frac{|\sinh z|}{|z|} - \frac{|\sinh z_0|}{|z_0|}\right| \\
&\geq \lambda\left|\frac{|z_0|\,|\sinh z| - |z|\,|\sinh z_0|}{|z_0|\,|z|}\right| \\
&\geq K(x, x_0)\left|\frac{|\sinh z| - |\sinh z_0|}{|z_0|\,|z|}\right| \quad (5.2.11)
\end{aligned}
$$

where, $K(x, x_0) = \lambda \min\{|z|, |z_0|\} > 0$. Now, if $a \leq x < x_0$, $|\sinh z| < |\sinh z_0|$ and if $x_0 < x \leq b$, $|\sinh z_0| < |\sinh z|$. Thus $||\sinh z_0| - |\sinh z|| > 0$ if $\Re(z) \in [a, x_0) \cup (x_0, b]$. Further, $|h_\lambda(z) - h_\lambda(z_0)| > 0$ for $\Re(z) = x_0 = \Re(z_0)$. Consequently, by (5.2.11), $|h_\lambda(z) - h_\lambda(z_0)| > 0$ for any $z \in \Gamma_{2,c}$. Since $z_0 \in \Gamma_{2,c+2\pi}$ is arbitrary, it follows that $h_\lambda(\Gamma_{2,c})$ and $h_\lambda(\Gamma_{2,c+2\pi})$ do not intersect. Further, by Proposition 5.2.2(b), $|\,h_\lambda\,(x + ic)|$ and $|\,h_\lambda\,(x + i(c + 2\pi))|$ are increasing functions of x, for $x \geq 2$. Thus, $h_\lambda(z)$ is one-to-one on $\Gamma_{2,c} \cup \Gamma_{2,c+2\pi}$ (See Figure 5.1).

As in the proof of Proposition 4.2.4, it follows that $h_\lambda(z)$ is one-to-one on any closed rectangle $R_{a,b,c}$. If $R_{a,b,c} = \{z = x + iy : a \leq x \leq b < -2, c \leq y \leq c + 2\pi\}$ is the closed rectangle contained in H_2^-, $R^*_{a,b,c} = \{-z : z \in R_{a,b,c}\}$ is contained in H_2^+ and it follows that

$h_\lambda(z)$ is one-to-one on the closed rectangle $R^*_{a,b,c}$. Therefore, $h_\lambda(z)$ is one-to-one on the closed rectangle $R_{a,b,c} \subseteq H_2$. This proves (c). □

The following proposition exhibits the characteristic property of the function $h_\lambda(z)$ for $\mu > 1$ to expand certain neighborhoods $U \subseteq D_\delta(z_0)$ of the point z_0 in such a way that $h_\lambda(z)$ is a homeomorphism from U to $D_{\mu\delta}(h_\lambda(z_0))$. Here, $D_r(z)$ denotes the open disk centered at z and having radius r.

Proposition 5.2.3. *Let $h_\lambda \in \mathcal{H}$ and $|h'_\lambda(z)| > \mu > 1$ for all $z \in D_\delta(z_0) \subseteq H_2$ where $\delta \leq \pi$ and $H_2 = \{z : \Re(z) \geq 2\} \cup \{z : \Re(z) \leq -2\}$. Then, there exists an open set $U \subseteq D_\delta(z_0)$ such that $h_\lambda : U \to D_{\mu\delta}(h_\lambda(z_0))$ is a homeomorphism.*

Proof. By proceeding on the same lines of the proof of Corollary 4.2.1, and using Proposition 5.2.2(c), the proposition follows immediately from Proposition 4.2.5. □

Proposition 5.2.4. *Let $h_\lambda \in \mathcal{H}$ and U be an open set containing z_0. Let $z_n = h^n_\lambda(z_0)$, $n = 1, 2, 3, \cdots$. Define $D = \left\{z \in \mathbb{C} : |h'_\lambda(z)| > \sqrt{2} \text{ and } |\Re(z)| \geq 2\right\}$. Suppose $D_{\sqrt{2}\pi}(z_n) \subset D$ for $n = 0, 1, 2, \cdots$. Let $S_{2\pi}(z_n)$ be the interior of the square with center at z_n and having sides of length 2π, parallel to the coordinate axes. Then, there exists an integer $N > 0$ and open sets $U_n \subseteq U$ for $n > N$, such that $h^n_\lambda : U_n \to S_{2\pi}(z_n)$ is a homeomorphism.*

Proof. The proof of Proposition 5.2.4 is same as that of the proof of Proposition 4.2.6 except for repeated application of Proposition 5.2.3 instead of Corollary 4.2.1. □

Proposition 5.2.5. *Let $h_\lambda \in \mathcal{H}$ and $Esc(h_\lambda) = clo\ \{z \in \mathbb{C} : h^n_\lambda(z) \to \infty\}$ be the closure of the set of escaping points of $h_\lambda(z)$. Suppose $z_0 \in Esc(h_\lambda)$ and U is any open set containing z_0. Then, there exist an integer $N > 0$ and points $z_1, z_2 \in U$ such that $h^N_\lambda(z_1)$ is a real number and $h^N_\lambda(z_2)$ is a purely imaginary number.*

Proof. Let $z_0 \in Esc(h_\lambda)$ and U be any neighborhood of z_0. Then, either $h^n_\lambda(z_0) \to \infty$ as $n \to \infty$ or there exists a point $\tilde{z} \in U$ such that $h^n_\lambda(\tilde{z}) \to \infty$ as $n \to \infty$. In the latter

case, rename $\tilde{z}$ as z_0, so that without loss of generality, if $z_0 \in Esc(h_\lambda)$, it may be assumed that $h_\lambda^n(z_0) \to \infty$ as $n \to \infty$.

For any $z = x + iy \neq 0$, the absolute value of $h_\lambda(z)$ is given by

$$|h_\lambda(x+iy)| = \lambda \left(\frac{\sin^2(y) + \sinh^2(x)}{x^2 + y^2} \right)^{1/2} .$$

It is easily seen that

$$|h_\lambda(x_0 + iy)| \to 0 \quad \text{as} \quad |y| \to \infty, \text{ for any fixed } x_0 \in \mathbb{R}$$

$$|h_\lambda(x + iy_0)| \to \infty \quad \text{as} \quad |x| \to \infty, \text{ for any fixed } y_0 \in \mathbb{R} .$$

Since $h_\lambda^n(z_0) \to \infty$ as $n \to \infty$, there exists a positive integer N_0 such that $h_\lambda^{N_0}(z_0) = w_0$ and $D_{\sqrt{2}\pi}(h_\lambda^n(w_0)) \subseteq D = \left\{z \in \mathbb{C} \ : \ |h'_\lambda(z)| > \sqrt{2} \text{ and } |\Re(z)| \geq 2\right\}$ for all $n = 0, 1, 2, \cdots$. Let V be a neighborhood of w_0 such that $V \subseteq h_\lambda^{N_0}(U)$. Adopting the lines of the proof of Proposition 4.2.7 and using Propositions 5.2.4 to the point w_0 and V, it follows that there exists an integer $n_0 > 0$ such that, if $n > n_0$, there exists an open set $V_n \subset V$ for which $h_\lambda^n : V_n \to S_{2\pi}(h_\lambda^n(w_0))$ is a homeomorphism. Further proceeding on the lines of the proof of Proposition 4.2.7 and using Proposition 5.2.2(a) and (c), for $n_1 > n_0$, $h_\lambda^{n_1+1}(V_n) \cap \mathbb{R} \neq \emptyset$ and $h_\lambda^{n_1+1}(V_n) \cap i\,\mathbb{R} \neq \emptyset$ follows. Therefore, there exists a point $\tilde{z}_1 \in V_n \subseteq V$ such that $h_\lambda^{n_1+1}(\tilde{z}_1)$ is a real number. By setting $z_1 = h_\lambda^{-N_0}(\tilde{z}_1)$, it follows that $h_\lambda^{N_0+n_1+1}(z_1)$ is a real number for $z_1 \in U$. Similarly, there exists a point $\tilde{z}_2 \in V_n \subseteq V$ such that $h_\lambda^{n_1+1}(\tilde{z}_2)$ is a purely imaginary number and by setting $z_2 = h_\lambda^{-N_0}(\tilde{z}_2)$, it follows that $h_\lambda^{N_0+n_1+1}(z_2)$ is a purely imaginary number for $z_2 \in U$. □

5.3 Bifurcation in the dynamics of $h_\lambda(x)$ for $x \in \mathbb{R}$

The dynamics of $h_\lambda(x) = \lambda \sinh x / x$ for $x \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ is investigated in this section. We find that bifurcation in the dynamics of $h_\lambda(x)$ occurs at $|\lambda| = \lambda^{**}$ ($\approx$ 1.104) where $\lambda^{**} = (x^{**})^2/\sinh x^{**}$ and x^{**} is the unique positive real root of the equation $\tanh x = x/2$.

Let $\Phi(x) = \sinh x/x$ for $x \in$ R $\setminus \{0\}$ and $\Phi(0) = 1$. It is easily seen that for $x > 0$ the functions $\Phi(x)$ and $\Phi'(x)$ are strictly increasing positive valued functions. Therefore, the function $\phi(x) = \Phi(x) - x\Phi'(x)$, is strictly decreasing in the interval $[0, \infty)$. Since $\Phi(x)$ is an even function, the function $\phi(x)$ is also an even function. Consequently, since $\Phi(0) = 1$ and $\Phi(2) < 0$, there exists a unique $x^{**} \in (0, 2)$ such that

$$\phi(x) \begin{cases} > 0 & \text{for } |x| < x^{**} \\ = 0 & \text{for } |x| = x^{**} \\ < 0 & \text{for } |x| > x^{**} \end{cases} \tag{5.3.1}$$

Throughout in the sequel, we denote

$$\lambda^{**} = \frac{1}{\Phi'(x^{**})} \tag{5.3.2}$$

where, x^{**} is the unique positive real root of the equation $\phi(x) = 0$.

The existence and the nature of fixed points on the real line for the function $h_\lambda(x)$ is investigated in the following theorem:

Theorem 5.3.1. *Let $h_\lambda(x) = \lambda \sinh x/x$ for $x \in$ R and λ be a non-zero real parameter.*

(a) *If $0 < \lambda < \lambda^{**}$, $h_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ (say) with $0 < a_\lambda < r_\lambda$.*

(b) *If $\lambda = \lambda^{**}$, $h_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^{**}$.*

(c) *If $\lambda > \lambda^{**}$, $h_\lambda(x)$ has no fixed points.*

(d) *If $-\lambda^{**} < \lambda < 0$, $h_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ (say) with $r_\lambda < a_\lambda < 0$.*

(e) *If $\lambda = -\lambda^{**}$, $h_\lambda(x)$ has a unique rationally indifferent fixed point at $x = -x^{**}$.*

(f) *If $\lambda < -\lambda^{**}$, $h_\lambda(x)$ has no fixed points.*

Proof. Define $g_\lambda(x) = h_\lambda(x) - x = \lambda\Phi(x) - x$ for $x \in$ R . The zeros of $g_\lambda(x)$ are fixed points of $h_\lambda(x)$. Further,

(i) $g_\lambda(x)$ is continuously differentiable in R. For $|x|$ sufficiently large $g_\lambda(x)$ is positive if $\lambda > 0$, and is negative if $\lambda < 0$.

(ii) $g_\lambda(x)$ has a unique local minimum at $\tilde{x} \equiv \tilde{x}(\lambda) > 0$ for $\lambda > 0$ and it has a unique local maximum at $\tilde{x} \equiv \tilde{x}(\lambda) < 0$ for $\lambda < 0$.

The assertion (i) is easily seen to hold. To see that (ii) holds, when $\lambda > 0$ observe that the function $g'_\lambda(x)$ is strictly increasing, $g'_\lambda(x) \to -\infty$ as $x \to -\infty$, $g'_\lambda(x) \to \infty$ as $x \to \infty$ and $g'_\lambda(0) = -1$. Therefore, due to continuity of $g'_\lambda(x)$, there exist a unique real number $\tilde{x} \equiv \tilde{x}(\lambda) > 0$ such that $g'_\lambda(\tilde{x}) = 0$, $g'_\lambda(x) < 0$ for $x < \tilde{x}$ and $g'_\lambda(x) > 0$ for $x > \tilde{x}$. Thus, in view of $g''_\lambda(\tilde{x}) > 0$, $g_\lambda(x)$ attains a local minimum value at $x = \tilde{x}$. When $\lambda < 0$, the function $g'_\lambda(x)$ is strictly decreasing, $g'_\lambda(x) \to \infty$ as $x \to -\infty$, $g'_\lambda(x) \to -\infty$ as $x \to \infty$ and $g'_\lambda(0) = -1$. Therefore, due to continuity of $g'_\lambda(x)$, there exist a unique real number $\tilde{x} \equiv \tilde{x}(\lambda) < 0$ such that $g'_\lambda(\tilde{x}) = 0$, $g'_\lambda(x) > 0$ for $x < \tilde{x}$ and $g'_\lambda(x) < 0$ for $x > \tilde{x}$. Thus, in view of $g''_\lambda(\tilde{x}) < 0$, $g_\lambda(x)$ attains a local maximum value at $x = \tilde{x}$.

(a) case: $0 < \lambda < \lambda^{**}$

Clearly, $g'_\lambda(\tilde{x}) = 0$ implies that $\lambda = 1/\Phi'(\tilde{x})$. Since $\lambda^{**} = 1/\Phi'(x^{**})$ and $\Phi'(x)$ is strictly increasing function, it follows that $\tilde{x} > x^{**}$ for $0 < \lambda < \lambda^{**}$. Thus, by (5.3.1), $\phi(\tilde{x}) = \Phi(\tilde{x}) - \tilde{x}\Phi'(\tilde{x}) = \Phi'(\tilde{x})\, g_\lambda(\tilde{x}) < 0$. Since $\Phi'(\tilde{x}) > 0$, $g_\lambda(\tilde{x}) < 0$ (See Figure 5.2(a)). Now, (i) and (ii) together with $g_\lambda(\tilde{x}) < 0$ imply that $g_\lambda(x)$ has only two zeros a_λ and r_λ (say) with $a_\lambda < \tilde{x} < r_\lambda$. Thus,

$$g_\lambda(x) \begin{cases} > 0 & \text{for } x \in (-\infty,\, a_\lambda) \cup (r_\lambda, \infty) \\ = 0 & \text{for } x = a_\lambda \text{ or } r_\lambda \\ < 0 & \text{for } x \in (a_\lambda,\, r_\lambda) \end{cases} \tag{5.3.3}$$

Since $0 < a_\lambda < \tilde{x} < r_\lambda$ implies that $g'_\lambda(a_\lambda) < 0$ and $g'_\lambda(r_\lambda) > 0$, it follows that $h'_\lambda(a_\lambda) < 1$ and $h'_\lambda(r_\lambda) > 1$. Thus, the point a_λ is an attracting fixed point and the point r_λ is a repelling fixed point of $h_\lambda(x)$. This proves (a).

(b) case: $\lambda = \lambda^{**}$

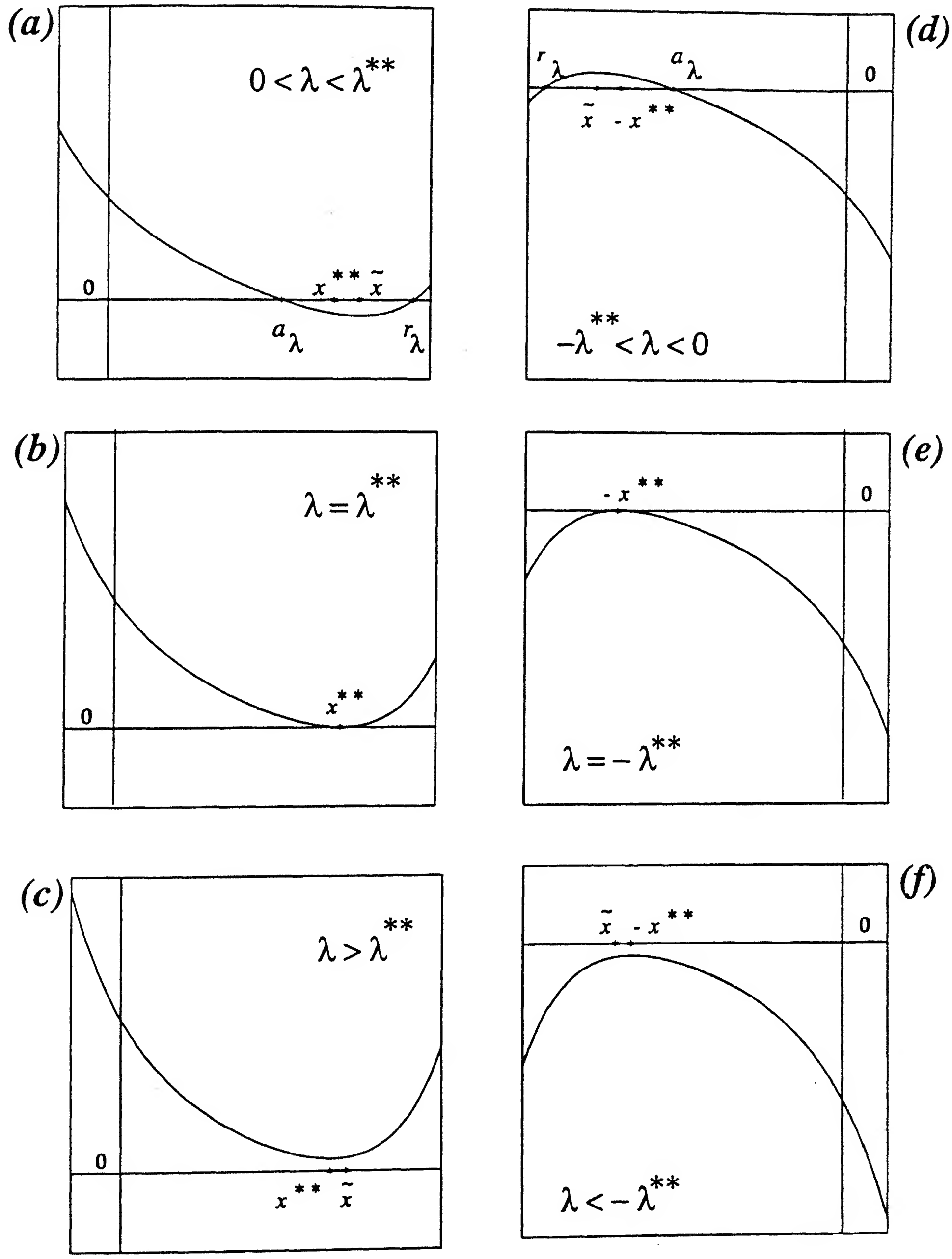


Figure 5.2: The graphs of $g_\lambda(x) = h_\lambda(x) - x$ for (a) $0 < \lambda < \lambda^*$, (b) $\lambda = \lambda^*$ (c) $\lambda > \lambda^*$, (d) $-\lambda^* < \lambda < 0$, (e) $\lambda = -\lambda^*$ and (f) $\lambda < -\lambda^*$.

If $\lambda = \lambda^{**}$ then $\tilde{x} = x^{**}$, $g_\lambda\,(x^{**}) = 0$ and $g'_\lambda\,(x^{**}) = 0$ (See Figure 5.2(b)). Consequently, $h'_{\lambda^{**}}(x^{**}) = 1$. Thus, in view of (ii),

$$g_\lambda(x) \begin{cases} > 0 & \text{for } x \neq x^{**} \\ = 0 & \text{for } x = x^{**} \end{cases} \tag{5.3.4}$$

Therefore, $h_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^{**}$.

This proves (b).

(c) case: $\lambda > \lambda^{**}$

If $\lambda > \lambda^{**}$ then $\tilde{x} < x^{**}$, $\phi(\tilde{x}) > 0$ and $g_\lambda\,(\tilde{x}) > 0$ (See Figure 5.2(c)). Consequently, in view of (ii), $g_\lambda(x) > g_\lambda\,(\tilde{x}) > 0$ for all $x \in$ R and hence $g_\lambda(x)$ has no zeros. Thus, $h_\lambda(x)$ has no fixed points if $\lambda > \lambda^{**}$, completing the proof of (c).

(d) case: $-\lambda^{**} < \lambda < 0$

Since $-\lambda^{**} = \frac{1}{\Phi'(-x^{**})}$, $\lambda = \frac{1}{\Phi'(\tilde{x})}$ and $\Phi'(x)$ is strictly increasing function, it follows that $\tilde{x} < -x^{**} < 0$ for $-\lambda^{**} < \lambda < 0$. Thus, by (5.3.1), $\phi(\tilde{x}) = \Phi(\tilde{x}) - \tilde{x}\Phi'(\tilde{x}) = \Phi'(\tilde{x})\; g_\lambda\,(\tilde{x}) < 0$. Since $\Phi'(\tilde{x}) < 0$, $g_\lambda\,(\tilde{x}) > 0$ (See Figure 5.2(d)). Now, (i) and (ii) together with $g_\lambda\,(\tilde{x}) > 0$ imply that $g_\lambda(x)$ has only two zeros r_λ and a_λ (say) with $r_\lambda < \tilde{x} < a_\lambda < 0$. Thus,

$$g_\lambda(x) \begin{cases} < 0 & \text{for } x \in (-\infty,\, r_\lambda\,) \cup (\,a_\lambda\,, \infty) \\ = 0 & \text{for } x = \,a_\lambda\, \text{ or } r_\lambda \\ > 0 & \text{for } x \in (\,r_\lambda\,,\, a_\lambda\,) \end{cases} \tag{5.3.5}$$

Since $r_\lambda < \tilde{x} < a_\lambda < 0$, $g'_\lambda\,(\tilde{x}) = 1$ and $g'_\lambda\,(0) = -1$ implies that $g'_\lambda\,(\,r_\lambda\,) > 0$ and $g'_\lambda\,(\,a_\lambda\,) < 0$, it follows that $h'_\lambda\,(\,r_\lambda\,) > 1$ and $h'_\lambda\,(\,a_\lambda\,) < 1$. Thus, the point r_λ is a repelling fixed point and the point a_λ is an attracting fixed point of $h_\lambda(x)$. This proves (d).

(e) case: $\lambda = -\lambda^{**}$

If $\lambda = -\lambda^{**}$ then $\tilde{x} = -x^{**}$, $g_\lambda\,(-x^{**}) = 0$ and $g'_\lambda\,(-x^{**}) = 0$ (See Figure 5.2(e)). Consequently, $h'_{-\lambda^{**}}(-x^{**}) = 1$. Thus, in view of (ii),

$$g_\lambda(x) \begin{cases} < 0 & \text{for } x \neq -x^{**} \\ = 0 & \text{for } x = -x^{**} \end{cases} \tag{5.3.6}$$

Therefore, $h_{-\lambda^{**}}(x)$ has a unique rationally indifferent fixed point at $x = -x^{**}$.

This proves (e).

(f) **case:** $\lambda < -\lambda^{**}$

If $\lambda < -\lambda^{**}$ then $-x^{**} < \tilde{x}$, $\phi(\tilde{x}) > 0$ and $g_\lambda(\tilde{x}) < 0$ (See Figure 5.2(f)). Consequently, in view of (ii), $g_\lambda(x) \leq g_\lambda(\tilde{x}) < 0$ for all $x \in \mathsf{R}$ and hence $g_\lambda(x)$ has no zeros. Thus, $h_\lambda(x)$ has no fixed points if $\lambda < -\lambda^{**}$, completing the proof of (f). □

The dynamics of $h_\lambda \in \mathcal{H}$ on the real line is investigated in the following theorem:

Theorem 5.3.2. *Let $h_\lambda(x) = \lambda \sinh x/x$ for $x \in \mathsf{R}$ and λ be a non-zero real parameter.*

(a) *If $0 < \lambda < \lambda^{**}$, $h_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $|x| < r_\lambda$ and $h_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > r_\lambda$, where a_λ and r_λ are the attracting and the repelling fixed points of $h_\lambda(x)$ respectively.*

(b) *If $\lambda = \lambda^{**}$, $h_\lambda^n(x) \to x^{**}$ as $n \to \infty$ for $|x| < x^{**}$ and $h_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > x^{**}$, where x^{**}, given by (5.3.2), is the rationally indifferent fixed point of $h_\lambda(x)$.*

(c) *If $\lambda > \lambda^{**}$, $h_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathsf{R}$.*

(d) *If $-\lambda^{**} < \lambda < 0$, $h_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $|x| < -r_\lambda$ and $h_\lambda^n(x) \to -\infty$ as $n \to \infty$ for $|x| > -r_\lambda$, where a_λ and r_λ are the attracting and the repelling fixed points of $h_\lambda(x)$ respectively.*

(e) *If $\lambda = -\lambda^{**}$, $h_\lambda^n(x) \to -x^{**}$ as $n \to \infty$ for $|x| < x^{**}$ and $h_\lambda^n(x) \to -\infty$ as $n \to \infty$ for $|x| > x^{**}$, where x^{**} is given by (5.3.2) and $-x^{**}$ is the rationally indifferent fixed point of $h_\lambda(x)$.*

(f) *If $\lambda < -\lambda^{**}$, $h_\lambda^n(x) \to -\infty$ as $n \to \infty$ for all $x \in \mathsf{R}$.*

Proof. (a) If $0 < \lambda < \lambda^{**}$, by Theorem 5.3.1(a), it follows that $h_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ with $0 < a_\lambda < \tilde{x} < r_\lambda$. By (5.3.3), for $a_\lambda < x < r_\lambda$,

$$h_\lambda(x) - a_\lambda < x - a_\lambda . \tag{5.3.7}$$

If $0 \leq x < a_\lambda$ then, by the mean value theorem, $| h_\lambda(x) - a_\lambda | \leq h'_\lambda(c)|x - a_\lambda|$, where $0 \leq x < c < a_\lambda$. Since $h'_\lambda(0) = 0$, $h'_\lambda(a_\lambda) < 1$ and $h'_\lambda(x)$ is strictly increasing, it follows that $h'_\lambda(c) < 1$. Consequently, $| h_\lambda(x) - a_\lambda | < |x - a_\lambda|$ for $x < a_\lambda$. This inequality together with inequality (5.3.7) gives that for $0 \leq x < r_\lambda$ and $x \neq a_\lambda$, $| h_\lambda(x) - a_\lambda | < |x - a_\lambda|$. Thus, for $0 \leq x < r_\lambda$, $h^n_\lambda(x) \to a_\lambda$ as $n \to \infty$. Since $h_\lambda(-x) = h_\lambda(x)$, it follows that $h^n_\lambda(x) \to a_\lambda$ as $n \to \infty$. for $-r_\lambda < x \leq 0$. Further, if $x > r_\lambda$, $h^n_\lambda(x) \to \infty$ as $n \to \infty$, since $h_\lambda(x) > x$ and $h'_\lambda(x) > 1$ for $x > r_\lambda$. Again, in view of $h_\lambda(-x) = h_\lambda(x)$, $h^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x < -r_\lambda$. This completes the proof of (a).

(b) By Theorem 5.3.1(b), if $\lambda = \lambda^{**}$ then $h_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^{**}$. Since $h'_{\lambda^{*}}(x) < 1$ for $x < x^{**}$, $h'_{\lambda^{**}}(x^{**}) = 1$ and $h'_{\lambda^{**}}(x) > 1$ for $x > x^{**}$, it follows that $|h_{\lambda^{**}}(x) - x^{**}| < |x - x^{**}|$ for $0 \leq x < x^{**}$. Therefore, $h^n_{\lambda^{**}}(x) \to x^{**}$ as $n \to \infty$ for $0 \leq x < x^{**}$. If $x > x^{**}$ then $h^n_{\lambda^{**}}(x) \to \infty$ as $n \to \infty$, since by (5.3.4), $h_{\lambda^{**}}(x) > x$ for $x > x^{**}$. Since $h_\lambda(x)$ is an even function, it follows that $h^n_{\lambda^{**}}(x) \to x^{**}$ as $n \to \infty$, for $-x^{**} < x \leq 0$ and $h^n_{\lambda^{**}}(x) \to \infty$ as $n \to \infty$ for $x < -x^{**}$. This proves (b).

(c) If $\lambda > \lambda^{**}$ then, for $x \in \mathrm{R}$, $h^n_\lambda(x) \to \infty$ as $n \to \infty$, since $h_\lambda(x) > x$ for $\lambda > \lambda^{**}$, completing the proof of (c).

(d) If $-\lambda^{**} < \lambda < 0$, by Theorem 5.3.1(d), it follows that $h_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ with $r_\lambda < \tilde{x} < a_\lambda < 0$. Therefore, by (5.3.5), for $r_\lambda < x < a_\lambda < 0$,

$$|h_\lambda(x) - a_\lambda| < |x - a_\lambda|. \tag{5.3.8}$$

If $a_\lambda < x \leq 0$ then, by the mean value theorem, $| h_\lambda(x) - a_\lambda | \leq h'_\lambda(c)|x - a_\lambda|$, where $a_\lambda < c < x \leq 0$. Since $h'_\lambda(0) = 0$, $h'_\lambda(a_\lambda) < 1$ and $h'_\lambda(x)$ is strictly decreasing, it follows that $h'_\lambda(c) < 1$. Consequently, $| h_\lambda(x) - a_\lambda | < |x - a_\lambda|$ for $x < a_\lambda$. This inequality together with inequality (5.3.8) gives that for $r_\lambda < x \leq 0$ and $x \neq a_\lambda$,

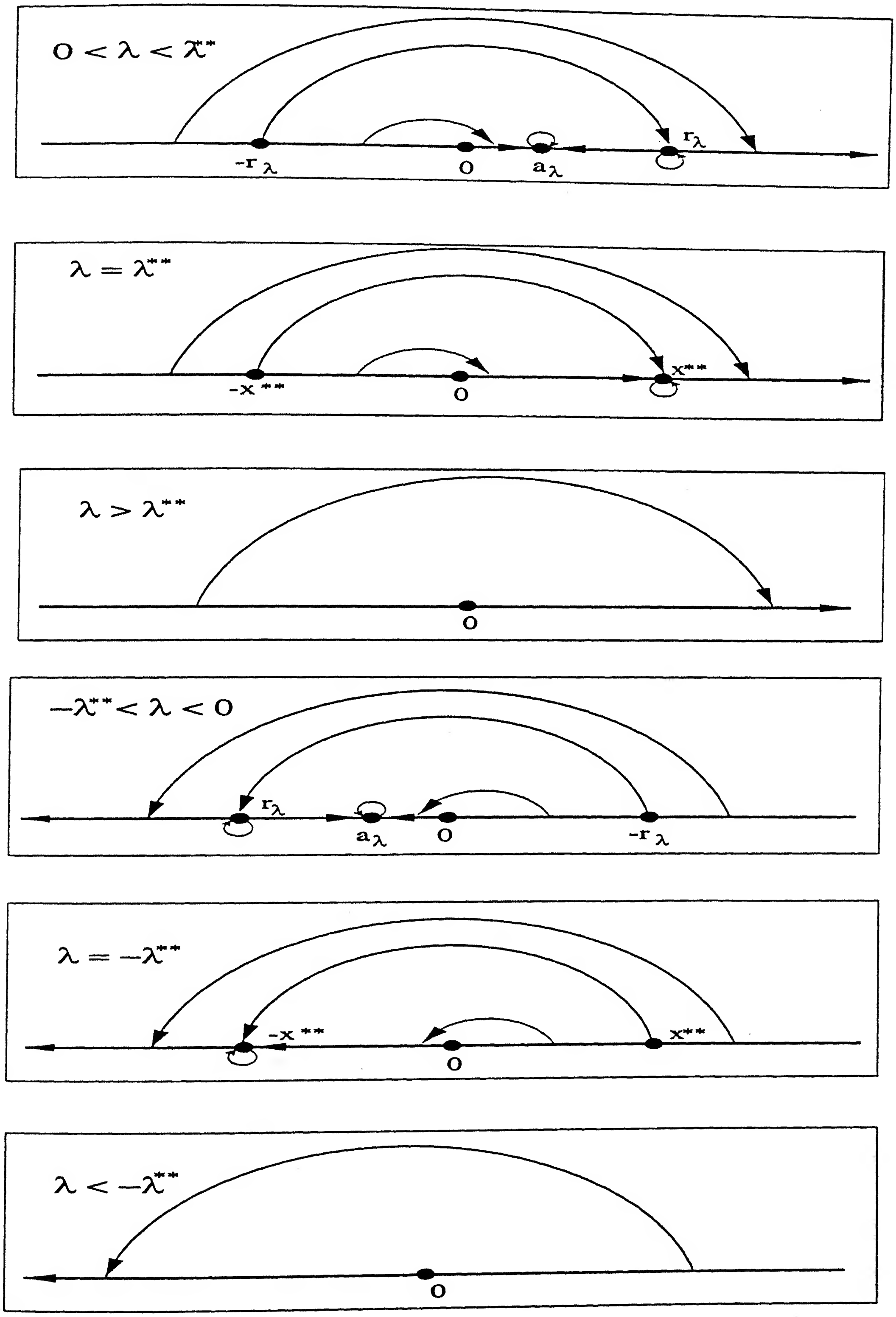


Figure 5.3: Phase portrait of the function $h_\lambda(x) = \lambda \sinh x/x$; ($\lambda \in$ R $\setminus \{0\}$).

$|\ h_\lambda(x)\ -\ a_\lambda\ | < |x -\ a_\lambda\ |$. Thus, for $r_\lambda\ < x \le 0$, $h_\lambda^n(x) \to\ a_\lambda$ as $n \to \infty$. Since $h_\lambda(-x) = h_\lambda(x)$, it follows that $h_\lambda^n(x) \to a_\lambda$ as $n \to \infty$. for $0 \le x < -\ r_\lambda$. Further, if $x <\ r_\lambda$, $h_\lambda^n(x) \to\ -\infty$ as $n \to \infty$, since $h_\lambda(x)\ < x$ for $x <\ r_\lambda$. Again, in view of $h_\lambda(-x) = h_\lambda(x)$, $h_\lambda^n(x) \to\ -\infty$ as $n \to \infty$ for $x > -\ r_\lambda$. This completes the proof of (d).

(e) By Theorem 5.3.1(e), if $\lambda = -\lambda^{**}$ then $h_\lambda(x)$ has a unique rationally indifferent fixed point at $x = -x^{**}$. Since $h'_{-\lambda^{**}}(x) < 1$ for $x \in (-x^{**}, 0]$, $h'_{-\lambda^{**}}(-x^{**}) = 1$ and $h'_{-\lambda^{**}}(x) > 1$ for $x < -x^{**}$, it follows that $|h_{-\lambda^{**}}(x) - (-x^{**})| < |x - (-x^{**})|$ for $x \in (-x^{**}, 0]$, Therefore, $h^n_{-\lambda^{**}}(x) \to -x^{**}$ as $n \to \infty$, for $x \in (-x^{**}, 0]$. If $x < -x^{**}$ then $h^n_{-\lambda^{**}}(x) \to -\infty$ as $n \to \infty$, since by (5.3.6), $h_{-\lambda^{**}}(x) < x$ for $x < -x^{**}$. Since $h_{-\lambda^{**}}(x)$ is an even function, it follows that $h^n_{-\lambda^{**}}(x) \to -x^{**}$ as $n \to \infty$, for $0 \le x < x^{**}$ and $h^n_{-\lambda^{**}}(x) \to -\infty$ as $n \to \infty$ for $x > x^{**}$. This proves (e).

(f) If $\lambda < -\lambda^{**}$ then, for $x \in \mathrm{R}$, $h_\lambda^n(x) \to\ -\infty$ as $n \to \infty$, since $h_\lambda(x) < 0$ and $|\ h_\lambda(x)\ | > |x|$ for $\lambda < -\lambda^{**}$, completing the proof of (f). □

Let

$$\hat{x_\lambda} = \begin{cases} r_\lambda & \text{if } 0 < \lambda < \lambda^{**} \\ x^{**} & \text{if } |\lambda| = \lambda^{**} \\ -\ r_\lambda & \text{if } -\lambda^{**} < \lambda < 0 \end{cases}$$

If $0 < \lambda \le \lambda^{**}$, it follows from Theorem 5.3.2 that under iteration of h_λ the orbits of all the points in absolute value less than $\hat{x_\lambda}$ remain bounded and the orbits of all the points in absolute value greater than $\hat{x_\lambda}$ become unbounded; while, if $\lambda > \lambda^{**}$, there is no real point whose orbit remains bounded under iteration of h_λ. Thus, bifurcation in the dynamics of $h_\lambda(x)$, $x \in \mathrm{R}$, occurs at the parameter value $\lambda = \lambda^{**}$. Similarly, if $\lambda < -\lambda^{**}$ there is no point whose orbit remain bounded, while $-\lambda^{**} \le \lambda < 0$, it follows from Theorem 5.3.2 that under iteration of h_λ the orbits of all the points in absolute value less than $\hat{x_\lambda}$ remain bounded and the orbits of all the points in absolute value greater than $\hat{x_\lambda}$ become unbounded. Thus, bifurcation in the dynamics $h_\lambda(x)$ $x \in \mathrm{R}$, also occurs

at the parameter value $\lambda = -\lambda^{**}$. Since $\phi(x^{**}) = 0$, it follows that $\lambda^{**} = (x^{**})^2/\sinh x^{**}$ where x^{**} is the unique positive real root of the equation $\tanh x = x/2$. The numerical computation of the root x^{**} of the equation $\tanh x = x/2$ by the bisection method gives $x^{**} \approx 1.594$. Thus, by (5.3.2) the critical parameter $\lambda^{**} \approx 1.104$. The bifurcation diagram for the function $h_\lambda(x) = \lambda \sinh x/x$ for $\lambda > 0$ is shown in Figure 5.3

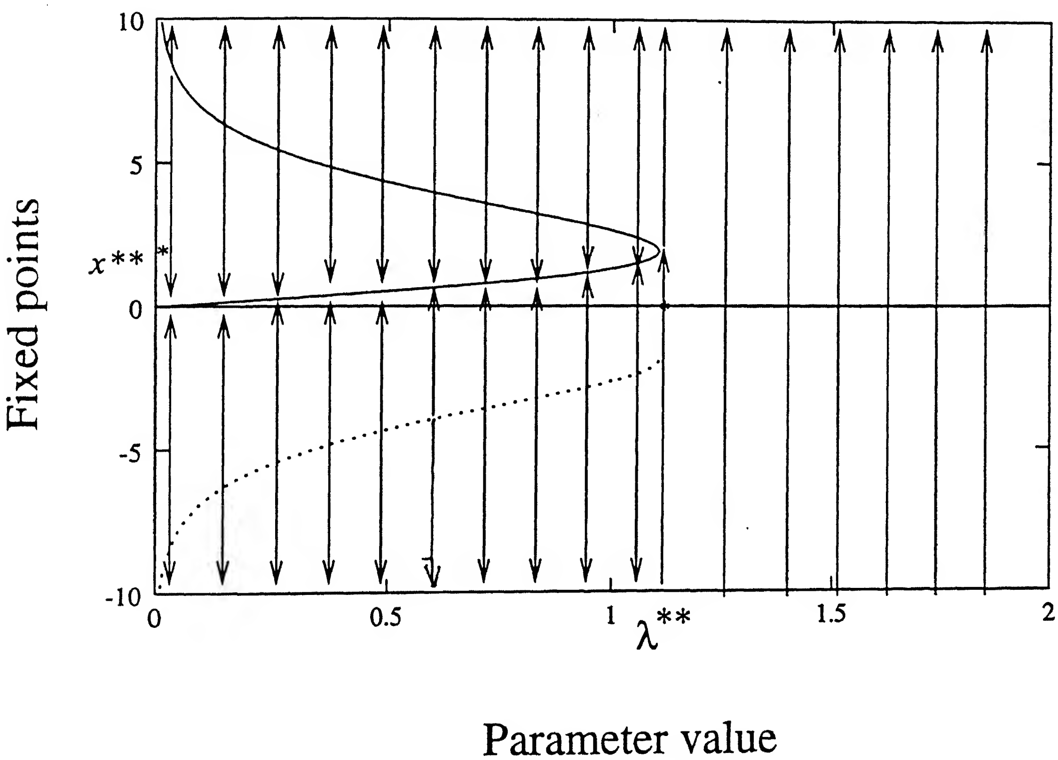


Figure 5.4: Bifurcation diagram for the function $h_\lambda(x) = \lambda \sinh x/x$ for $\lambda > 0$.

5.4 Dynamics of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < |\lambda| \leq \lambda^{**} \approx 1.104$

In this section, we first study the dynamics $h_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < |\lambda| < \lambda^{**}$, where λ^{**} is defined by (5.3.2).

If $0 < |\lambda| < \lambda^{**}$, by Theorem 5.3.1(a)and (d) it follows that $h_\lambda(z)$ has a real attracting fixed point a_λ and a real repelling fixed point r_λ such that $\tilde{x}$ with $h'_\lambda(\tilde{x}) = 1$ satisfies

$0 < |\, a_\lambda \,| < |\tilde{x}| < |\, r_\lambda \,|$. Let

$$\mathsf{A}(a_\lambda) = \{z \in \mathbb{C} : h_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty \}.$$

be the basin of attraction of the attracting fixed point a_λ of $h_\lambda(z)$ for $0 < |\lambda| < \lambda^{**}$. By Theorem 5.3.1(a), $h_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $|x| < |\, r_\lambda \,|$ and $|h_\lambda^n(x)| \to \infty$ as $n \to \infty$ for $|x| > |\, r_\lambda \,|$. Clearly, $\mathsf{A}(a_\lambda)$ contains the interval $(-|\, r_\lambda \,|, |\, r_\lambda \,|)$.

We prove in Proposition 5.4.1 below that the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point a_λ of $h_\lambda(z)$ for $0 < |\lambda| < \lambda^{**}$ contains the set $D = \{z : |h_\lambda(z)| < |\tilde{x}|\}$. Theorem 5.4.1 shows that the Julia set $\mathcal{J}(h_\lambda)$ is the closure of the set of escaping points of $h_\lambda(z)$ for $0 < |\lambda| < \lambda^{**}$, thus rendering a computationally useful algorithm to generate the pictures of the Julia set of $h_\lambda(z)$. Further in Theorem 5.4.2, the Julia set $\mathcal{J}(h_\lambda)$ is seen also to be the complement of the basin of attraction $\mathsf{A}(a_\lambda)$ of the attracting real fixed point of $h_\lambda(z)$. Finally, in Theorem 5.4.3, it is shown that, under a certain condition, the basin of attraction $\mathsf{A}(a_\lambda)$ of the attracting real fixed point of $h_\lambda(z)$, $0 < |\lambda| < \lambda^{**}$ is a dense subset of the complex plane.

Proposition 5.4.1. *Let $h_\lambda \in \mathcal{H}$ and $0 < |\lambda| < \lambda^{**}$. Then, the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point a_λ contains $D = \{z \in \mathbb{C} : |h_\lambda(z)| < |\tilde{x}|\}$, where $\tilde{x}$ is the real number such that $h'_\lambda(\tilde{x}) = 1$. Further, $|\tilde{x}| > x^{**} \approx 1.594$, where x^{**} is given by (5.3.2).*

Proof. Let $h_\lambda(z) = \lambda\Phi(z)$ for $z \in \mathbb{C}$, where $\Phi(z) = \sinh z/z$ and $\Phi(0) = \lambda$. If $0 < \lambda < \lambda^{**}$ then $\frac{1}{\Phi'(\tilde{x})} = \lambda < \lambda^{**} = \frac{1}{\Phi'(x^{**})}$. Since $\Phi'(x)$ is strictly increasing function, $\tilde{x} > x^{**} \approx 1.594$ follows. If $-\lambda^{**} < \lambda < 0$ then $-\lambda^{**} = \frac{-1}{\Phi'(x^{**})} = \frac{1}{\Phi'(-x^{**})} < \lambda = \frac{1}{\Phi'(\tilde{x})}$ and $\Phi'(x)$ is strictly increasing function it follows that $\tilde{x} < -x^{**} < 0$. Thus, $|\tilde{x}| > x^{**} \approx 1.594$. Adopting the lines of the proof of Proposition 3.3.1 and using the equation (5.3.1), it follows that $h_\lambda(z)$ maps the open disk $D_{\tilde{x}}(0)$, centered at origin and having radius $\tilde{x}$ into itself. The function $h_\lambda(z)$ has zeros only at $k\pi i$, for $k = 0, \pm 1, \pm 2, \cdots$ and $|h_\lambda(iy)| \leq |\lambda| < |\tilde{x}|$ for $y \in \mathbb{R}$.

Since all the zeros of $h_\lambda(z)$ lie in the imaginary axis it follows that ([74], vol.1. p376) the curve $\gamma = \{z \in \mathbb{C} : \Re(z) > 0$ and $|h_\lambda(z)| = |\tilde{x}|\}$ is connected and not self intersecting. Therefore $D = \{z \in \mathbb{C} : |h_\lambda(z)| < |\tilde{x}|\}$ is a simply connected domain. Since $h_\lambda(z)$ maps D into $D_{\tilde{x}}(0)$ and $h_\lambda(D_{\tilde{x}}(0)) \subseteq D_{\tilde{x}}(0)$, by Schwarz lemma ([30], p264), $h_\lambda^n(z) \to a_\lambda$ as $n \to \infty$ for all $z \in D$. Thus, $A(a_\lambda) \supset D$. □

Remark 5.4.1. *(i) The function $h_\lambda(z)$ has only one finite asymptotic value, namely 0. Since the forward orbit of the asymptotic value 0 is attracted by the attracting fixed point a_λ of $h_\lambda(z)$ for $0 < |\lambda| < \lambda^{**}$, the basin of attraction $A(a_\lambda)$ is unbounded.*
(ii) Since $|h_\lambda(ix)| < |\lambda|$ for all $x \in \mathbb{R}$, it follows by Proposition 5.4.1 that the imaginary axis is contained in $A(a_\lambda)$.

By Theorem 5.3.1(a) and (d), for $0 < |\lambda| < \lambda^{**}$, r_λ is the repelling fixed point for $h_\lambda(z)$ and therefore r_λ belongs to the Julia set $\mathcal{J}(h_\lambda)$ of $h_\lambda(z)$. Theorem 5.3.2(a) and (d), gives that all the points $|x| > |r_\lambda|$ are escaping points of $h_\lambda(z)$. In the following theorem, we find that all points $|x| > |r_\lambda|$ belong to the Julia set $\mathcal{J}(h_\lambda)$. In fact, a characterization of the Julia set of $h_\lambda(z)$ as the closure of the set of all escaping points of $h_\lambda(z)$ is found in Theorem 5.4.1.

Theorem 5.4.1. *Let $Esc(h_\lambda) = clo\ \{z \in \mathbb{C} : h_\lambda^n(z) \to \infty$ as $n \to \infty\}$ be the closure of the set of escaping points of $h_\lambda(z)$. If $0 < |\lambda| < \lambda^{**}$ then, the Julia set $\mathcal{J}(h_\lambda) = Esc(h_\lambda)$.*

Proof. Let $z_0 \in \mathcal{J}(h_\lambda)$ and U be any neighborhood of z_0. Since $\{h_\lambda^n\}$ is not normal in any neighborhood of z_0, by Montel's theorem ([76], c.f. Theorem 1.1.1), $\bigcup_n \{h_\lambda^n(U)\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > |r_\lambda|$ such that $\hat{x} \in \bigcup_n \{h_\lambda^n(U)\}$, where r_λ is the repelling fixed point of $h_\lambda(z)$. Therefore, there exists a point $\hat{z} \in U$ such that $h_\lambda^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Theorem 5.3.2(a) and (d), it follows

that $|h_\lambda^n(\hat{x})| \to \infty$ as $n \to \infty$, since $\hat{x} > | r_\lambda |$. Thus, there exists a point $\hat{z} \in U$ such that $|h_\lambda^n(\hat{z})| \to \infty$ as $n \to \infty$ and $z_0 \in \mathrm{Esc}(h_\lambda)$ follows. Consequently, $\mathcal{J}(h_\lambda) \subseteq \mathrm{Esc}(h_\lambda)$.

To prove that $\mathrm{Esc}(h_\lambda) \subseteq \mathcal{J}(h_\lambda)$, we note that if $z_0 \in \mathrm{Esc}(h_\lambda)$, then $z_0 \notin \mathsf{A}(a_\lambda)$. Let U be any neighborhood of z_0. By Proposition 5.2.5, there exist an integer $N > 0$ and a point $\tilde{z} \in U$ such that $h_\lambda^N(\tilde{z}) = i\tilde{y}$, $\tilde{y} \in \mathsf{R}$. It therefore follows from Proposition 5.4.1 that $\tilde{z} \in \mathsf{A}(a_\lambda)$. Thus, in the neighborhood of z_0, there exists a point $\tilde{z} \in \mathsf{A}(a_\lambda)$. Now the orbit $\{h_\lambda^n(\tilde{z})\}$ of $\tilde{z}$ is bounded, while the orbit of z_0 escapes to ∞ under iteration of h_λ . Thus, $\{h_\lambda^n\}$ is not normal at z_0. Consequently, $z_0 \in \mathcal{J}(h_\lambda)$ and $\mathrm{Esc}(h_\lambda) \subseteq \mathcal{J}(h_\lambda)$ follows. $\square$

The following theorem shows that $\mathcal{F}(h_\lambda)$ has only one component which is a basin of attraction of an attracting fixed point of $h_\lambda(z)$ for $0 < |\lambda| < \lambda^{**}$.

Theorem 5.4.2. *Let $h_\lambda \in \mathcal{H}$ and $0 < |\lambda| < \lambda^{**}$. Then, $\mathcal{F}(h_\lambda) = \mathsf{A}(a_\lambda)$, where $\mathsf{A}(a_\lambda)$ is the basin of attraction of the real fixed point a_λ of $h_\lambda(z)$.*

Proof. If a point z_0 lies on an attracting cycle or a parabolic cycle of an entire transcendental function then the orbit of at least one of the singular values is attracted to the orbit of z_0 ([32], p182). Further, If U is a Siegel disk then the orbit of atleast one of the critical points is dense in the boundary of U ([32], p184). Since 0 is the only asymptotic value for $h_\lambda(z)$ and by Proposition 5.2.1, all critical values are real and bounded by λ in absolute value, it follows that all the singular values of $h_\lambda(z)$ lie in the interval $[-\lambda, \lambda]$. Since $|\lambda| < | r_\lambda |$, by Theorem 5.3.2(a) and (d), the orbits of all singular values lie in $\mathsf{A}(a_\lambda)$, where $\mathsf{A}(a_\lambda)$ is the basin of attraction of the real attracting fixed point a_λ of $h_\lambda(z)$. Consequently, all singular values of $h_\lambda(z)$ and their orbits lie in the same component of $\mathsf{A}(a_\lambda)$. Therefore, in view of Theorems 1.1.8 and 1.1.9, it follows that $h_\lambda(z)$ has no parabolic domains and no Siegel disks in $\mathcal{F}(h_\lambda)$. Further, $\mathcal{F}(h_\lambda)$ has no attracting basins other than $\mathsf{A}(a_\lambda)$. By Theorem 5.4.1, it follows that $h_\lambda(z)$ do not have domain at infinity. If U is a wandering

domain of $h_\lambda(z)$ then all the finite limit functions of $\{h_\lambda^n|_U\}$ are contained in the derived set of forward orbits of all singular values of $h_\lambda(z)$ ([20], c.f. Theorem 1.1.10). Let U_0 be one of the component of wandering domain and $w \in U_0$. Since $\{h_\lambda^n(z)\}$ is normal in some neighborhood $N(w)$ of w. The limit function of $\{h_\lambda^n(z)|_{N(w)}\}$ can not be infinite. If the limit function of $\{h_\lambda^n(z)|_{N(w)}\}$ is finite, then the limit function is contained in the derived set of forward orbits of all singular values. But the forward orbits of all singular values tend to the attracting fixed point $\mathsf{A}(a_\lambda)$. Therefore, the function $h_\lambda(z)$, $0 < |\lambda| < \lambda^{**}$ can not have a wandering domain. Therefore, the only possible stable component U of $\mathcal{F}(h_\lambda)$ is a basin of attraction of an attracting fixed point of $h_\lambda(z)$. □

Remark 5.4.2. *(i) By the above theorem, the basin of attraction* $\mathsf{A}(a_\lambda)$ *is a completely invariant set, since* $\mathcal{F}(h_\lambda)$ *is completely invariant.*
(ii) It follows from Theorem 5.4.2 *that* $\mathcal{J}(h_\lambda) = (\mathsf{A}(a_\lambda))^c$, *giving another characterization of the Julia set as the complement of basin of attraction of non-critically finite entire transcendental function* $h_\lambda(z)$ *for* $0 < |\lambda| < \lambda^{**}$.

The following theorem shows that if $\mathsf{A}(a_\lambda)$ contains $D^* = \{z \in \mathbb{C} : |h_\lambda'(z)| < 1\}$ then $\mathsf{A}(a_\lambda)$ is a dense subset of the complex plane.

Theorem 5.4.3. *Let* $h_\lambda \in \mathcal{H}$ *and* $D^* = \{z \in \mathbb{C} : |h_\lambda'(z)| < 1\}$ *be a proper subset of the basin of attraction* $\mathsf{A}(a_\lambda)$ *of the attracting real fixed point* a_λ *of* $h_\lambda(z)$ *for* $0 < |\lambda| < \lambda^{**}$. *Then,* $\mathsf{A}(a_\lambda)$ *is a dense subset of* $\mathbb{C}$.

Proof. Let $z_0 \in (A(a_\lambda))^c$ and U be any open set containing z_0. We claim that $U \cap \mathsf{A}(a_\lambda) \neq \emptyset$. If $U \cap \mathsf{A}(a_\lambda) = \emptyset$ then $h_\lambda^n(U) \cap \mathsf{A}(a_\lambda) = \emptyset$ for all n. Proceeding on the lines of proof similar to that Theorem 4.4.3, it follows that there is an open disk $D_\rho(h_\lambda^n(z_0))$ of radius $\rho = \mu_1\mu_2\cdots\mu_n\delta$, center $h_\lambda^n(z_0)$ and contained in $h_\lambda^n(U)$, that does not meet $\mathsf{A}(a_\lambda)$. If n is chosen large enough so that $\rho = \mu_1\mu_2\cdots\mu_n\delta \geq \pi$, then $D_\rho(h_\lambda^n(z_0))$ must contain atleast one vertical line segment of length 2π, say L. But, the line L is

mapped by $h_\lambda(z)$ to a curve which is starlike with respect to origin and cuts the imaginary axis by Proposition 5.4.1. Let $w_0 = h_\lambda(L) \cap i\,\mathbb{R} = iy_0$ (*say*). Then, by Remark 5.4.1(ii), it follows that $w_0 \in A(a_\lambda)$, which leads to a contradiction. Therefore, $U \cap A(a_\lambda) \neq \emptyset$ and so $A(a_\lambda)$ is a dense subset of $\mathbb{C}$. □

Remark 5.4.3. *Since* $A(a_\lambda)$ *is a dense subset of* $\mathbb{C}$ *the complement of* $A(a_\lambda)$ *is a nowhere dense subset of the complex plane. Thus, in view of Theorem* 5.4.2, *for* $0 < |\lambda| < \lambda^{**}$, *the Julia set of* $h_\lambda(z)$ *is a nowhere dense subset of the complex plane, if* $D^* = \{z \in \mathbb{C} : |h'_\lambda(z)| < 1\} \subset A(a_\lambda)$.

Remark 5.4.4. *The dynamics of* $h_\lambda \in \mathcal{H}$ *for* $|\lambda| = \lambda^{**}$ *is similar to that of the dynamics of* $h_\lambda(z)$ *for* $0 < |\lambda| < \lambda^{**}$ *except for having parabolic domain corresponding to the rationally indifferent fixed point in the Fatou set for* $|\lambda| = \lambda^{**}$ *instead of having basin of attraction as its Fatou set for* $0 < |\lambda| < \lambda^{**}$.

5.5 Dynamics of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| > \lambda^{**} \approx 1.104$

In this section, we describe the dynamics of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| > \lambda^{**}$. In this case, we mainly prove a result analogous to Theorem 4.4.1 that characterizes the Julia set of $h_\lambda(z)$, $|\lambda| > \lambda^{**}$, as the closure of the set of all escaping points. First we prove the following proposition giving that, for $|\lambda| > \lambda^{**}$, all the real points and the purely imaginary points are contained in the Julia set $\mathcal{J}(h_\lambda)$ of $h_\lambda(z)$.

Proposition 5.5.1. *For* $h_\lambda \in \mathcal{H}$, $|\lambda| > \lambda^{**}$, *both the coordiante axes* $\mathbb{R}$ *and* $i\,\mathbb{R}$ *are contained in the Julia set* $\mathcal{J}(h_\lambda)$.

Proof. Let $x_0 \in \mathbb{R} \setminus \{0\}$ and U be any open set containing x_0. Set $x_n = h_\lambda^n(x_0)$, $n = 1, 2, 3, \cdots$. By Theorem 5.3.2(c) and (f), it follows that $|x_n| \to \infty$ as $n \to \infty$. Since $h'_\lambda(x) \neq 0$, for all $x \in \mathbb{R} \setminus \{0\}$, $h_\lambda(x)$ is locally one-to-one for all $x \in \mathbb{R} \setminus \{0\}$. Further

$|\,h_\lambda(x)\,| > |x|$ for $x \in \mathbb{R}$ and $|\,h'_\lambda(x)\,| \to \infty$ as $|x| \to \infty$. Therefore, using the continuity of $h'_\lambda(z)$, there exist an integer $N_0 > 0$ and an open ball $D_\delta(x_0) \subseteq U$ such that

(i) $h_\lambda^{N_0} : D_\delta(x_0) \to V$ is a homeomorphism.

(ii) $D_{\sqrt{2}\,\pi}(x_m) \subset \left\{z \in \mathbb{C} \,:\, |h'_\lambda(z)| > \sqrt{2} \text{ and } |\Re(z)| \geq 2\right\}$ for $m \geq N_0$.

Proposition 5.2.4, applied to the point x_{N_0} and the open set V containing x_{N_0}, gives that there exists an integer $N_1 > 0$ such that, if $n > N_1$, there exists an open set $V_n \subseteq V$ for which $h_\lambda^n : V_n \to S_{2\pi}(h_\lambda^n(x_{N_0}))$ is a homeomorphism, where $S_{2\pi}(h_\lambda^n(x_{N_0}))$ is the interior of the square with center at $h_\lambda^n(x_{N_0})$ and having sides of length 2π, parallel to the co-ordinate axes. Since $|h'_\lambda(x_n)| \to \infty$ as $n \to \infty$, by Proposition 4.1.5, it follows that h_λ expands $S_{2\pi}(h_\lambda^n(x_{N_0}))$ with arbitrarily large scaling ratio as $n \to \infty$. Therefore, choose $N_2 > N_1$ large enough so that $h_\lambda(S_{2\pi}(h_\lambda^{n-1}(x_{N_0}))) \supset S_{2\pi}(h_\lambda^n(x_{N_0}))$ for $n > N_2$.

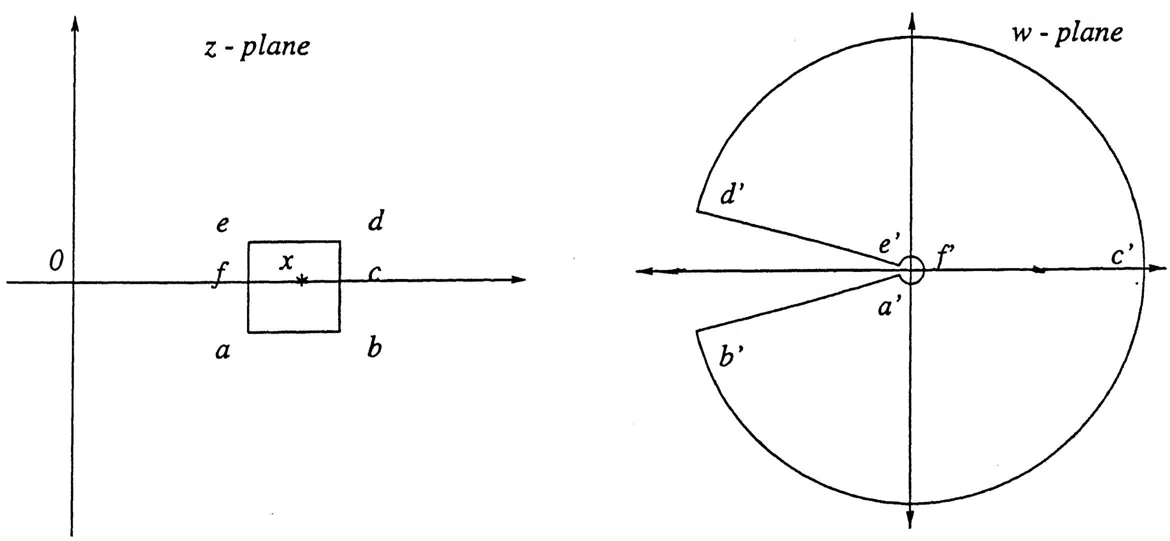


Figure 5.5: Image of the square $S_{2\pi}(x)$, under the mapping $w = h_\lambda(z)$.

Let $\Gamma : ab\widehat{c}def$ denote the boundary of the square $S_{2\pi}(h_\lambda^n(x_{N_0}))$ and $h_\lambda(\Gamma)$ be denoted by $a'b'\widehat{c'}d'e'f'$ (See Figure 5.5). Since the mappings $h_\lambda^{N_0} : D_\delta(x_0) \to V$, $h_\lambda^n : V_n \to S_{2\pi}(h_\lambda^n(x_{N_0}))$ for $n > N_1$ and $h_\lambda : S_{2\pi}(h_\lambda^n(x_{N_0})) \to h_\lambda(S_{2\pi}(h_\lambda^n(x_{N_0})))$ are homeomorphisms, for any point $w_1 \in h_\lambda(S_{2\pi}(h_\lambda^n(x_{N_0})))$ on the imaginary axis sufficiently near to the inner boundary, there

exists a point in U that gets mapped to w_1 by $h_\lambda^{N_0+n+1}$ for each $n > N_1$. Similarly, for any real point $w_2 \in h_\lambda(S_{2\pi}(h_\lambda^n(x_{N_0})))$ sufficiently near the boundary point c', there exists a point in U that gets mapped to w_2 by $h_\lambda^{N_0+n+1}$, for each $n > N_1$. The points w_1 and w_2, in turn, get mapped by the function $h_\lambda(z)$ respectively to the points $h_\lambda(w_1)$ and $h_\lambda(w_2)$ one inside the open disk $D_\lambda(0)$ and the other arbitrarily close to ∞. Therefore, for each integer $n > N_0 + N_1 + 2$, there exist distinct points x_1 and x_2 in the open set U containing x_0 that get mapped by the function $h_\lambda^n(z)$ to the points $h_\lambda^n(x_1)$ and $h_\lambda^n(x_2)$ one inside the open disk $D_\lambda(0)$ and the other arbitrarily close to ∞ respectively. Thus, the family of functions $\{h_\lambda^n\}$ is not normal in U and so $x_0 \in \mathcal{J}(h_\lambda)$. Since x_0 is any arbitrary non-zero real point, it follows that $\mathcal{J}(h_\lambda)$ contains $\mathbb{R} \setminus \{0\}$. Since $h_\lambda(0) = \lambda$ and $\lambda \in \mathcal{J}(h_\lambda)$ it follows that the point $x = 0$ lies in $\mathcal{J}(h_\lambda)$. Thus, the julia set of h_λ, $|\lambda| > \lambda^{**}$ contains the real line $\mathbb{R}$. Since the purely imaginary numbers ($i\,\mathbb{R}$) are preimages of the real numbers under the mapping h_λ, it follows that the purely imaginary numbers are contained in the julia set of h_λ for $|\lambda| > \lambda^{**}$. □

The following theorem, analogous to Theorem 5.4.1, provides a characterization for the Julia set of $h_\lambda(z)$ for $|\lambda| > \lambda^{**}$:

Theorem 5.5.1. *Let $h_\lambda \in \mathcal{H}$ and $Esc(h_\lambda) = clo\ \{z \in \mathbb{C} : h_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$ be the closure of the set of escaping points of $h_\lambda(z)$. If $|\lambda| > \lambda^{**}$, then the Julia set $\mathcal{J}(h_\lambda) = Esc(h_\lambda)$.*

Proof. The inclusion relation $\mathcal{J}(h_\lambda) \subseteq \mathrm{Esc}(h_\lambda)$ follows on the lines of proof similar to that of Theorem 5.4.1. We need only to prove $\mathrm{Esc}(h_\lambda) \subseteq \mathcal{J}(h_\lambda)$.

Let $z_0 \in \mathrm{Esc}(h_\lambda)$ and U be an open set containing z_0. By Proposition 5.2.5, it follows that there exists an integer $N > 0$ and a point $\hat{z} \in U$ such that $h_\lambda^N(\hat{z})$ is a real number. Therefore, by Proposition 5.5.1, $\hat{z} \in \mathcal{J}(h_\lambda)$. Thus, $\{h_\lambda^n\}$ is not normal in U. Consequently, $\mathrm{Esc}(h_\lambda) \subseteq \mathcal{J}(h_\lambda)$. This completes the proof of Theorem 5.5.1. □

5.6 Applications

In this section, we first generate the pictures of the Julia set of $h_\lambda(z)$ for various values of λ as an application of Theorems 5.4.1 and 5.5.1. Next, the results obtained in the previous sections for the dynamics of $h_\lambda \in \mathcal{H}$ are compared with those of Devaney and Durkin [33] obtained for the dynamics of the even and critically finite entire functions $C_\lambda(z) = \lambda\ i \cos z$, $\lambda \in \mathbb{R} \setminus \{0\}$. Further, the results on the dynamics of non-critically finite entire functions $h_\lambda(z)$ for $\lambda > 0$ found in the present chapter are compared with those on the dynamics of the non-critically finite entire functions $f_\lambda(z) = \lambda(e^z - 1)/z$ for $\lambda > 0$ found in Chapter 4.

In view of Theorems 5.4.1 and 5.5.1, the algorithm in Section 4.6.1 can be adopted to generate picture of Julia set $\mathcal{J}(h_\lambda)$. We choose $\lambda = 1.1 < \lambda^{**}$ and $\lambda = 1.2 > \lambda^{**}$. Consider the rectangular domain $R = \{z \in \mathbb{C} : -6 \leq \Re(z) \leq 6 \text{ and } -6 \leq \Im(z) \leq 6\}$. To generate these pictures, for each grid point in the rectangle R the maximum number of iterations $N = 240$ is allowed for a possible escape of the bound $M = 100$. The generated pictures of the Julia sets for $\lambda = 1.0$ and $\lambda = 1.2$ are shown in Figure 5.6.

It is found that the Julia set of $h_\lambda(z)$ for $\lambda = 1.1 < \lambda^{**}$ have the same pattern as those of the Julia sets of $h_\lambda(z)$ for all λ satisfying $0 < |\lambda| < \lambda^{**}$. Further, the nature of picture of the Julia set $\mathcal{J}(h_{1.1})$ remains unaltered by increasing the maximum number of iterations $N \geq 240$ for a fixed bound $M = 100$, while the nature of picture of the Julia set of $h_\lambda(z)$ for $\lambda = 1.2 > \lambda^{**}$ shows a distinct change on increasing the number of iterations and, for a fixed bound $M = 100$, it becomes increasingly more black as the maximum number of iterations N is increased. Figure 5.6 suggests that the Julia set of $h_\lambda(z)$ admits Cantor bouquets for $0 < \lambda < \lambda^{**}$ and there is an explosion in the Julia set of $h_\lambda(z)$ as λ crosses the threshold value λ^{**}.

Table 5.1 gives a comparison between the results obtained in this chapter for the dynam-

ical properties of non-critically finite function $h_\lambda(z)$ with those of Devaney and Durkin [33] obtained for the dynamics of critically finite even entire function $C_\lambda(z) = \lambda\, i \cos z$ where λ is a non-zero real parameter.

Finally, a comparison between the dynamical properties of non-critically finite even entire function $h_\lambda(z)$, $\lambda > 0$, as obtained in the present chapter, and those of non-critically finite entire function $f_\lambda(z) = \lambda(e^z - 1)/z$, $\lambda > 0$, as obtained in Chapter 4, is given in Table 5.2.

$h_\lambda(z) = \lambda \frac{\sinh z}{z}$, $\lambda \in \mathbb{R} \setminus \{0\}$	$C_\lambda(z) = \lambda\, i\, \cos z$, $\lambda \in \mathbb{R} \setminus \{0\}$
$h_\lambda(z)$ is not a periodic function.	$C_\lambda(z)$ is a periodic function with period 2π.
$h_\lambda(z)$ is an even function.	$C_\lambda(z)$ is an even function.
$h_\lambda(z)$ has infinitely many critical values.	$C_\lambda(z)$ has only two critical values, namely $\pm i\lambda$.
$h_\lambda(z)$ has only one (finite) asymptotic value, namely 0.	$C_\lambda(z)$ has no (finite) asymptotic values.
$\mathcal{J}(h_\lambda) = Esc(h_\lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$.	$\mathcal{J}(C_\lambda) = Esc(C_\lambda)$ for $\lambda \in \mathbb{R} \setminus \{0\}$.
The bifurcation occurs in the dynamics of $h_\lambda(z)$ at the critical parameter value $\pm\lambda^{**} \approx 1.104$, defined by (5.3.2).	The bifurcation occurs in the dynamics of $C_\lambda(z)$ at the critical parameter value $\pm\lambda^{*} \approx 0.66274$.
$\mathcal{J}(h_\lambda)$ contains both the real axis and the imaginary axis for $\|\lambda\| > \lambda^{**}$.	$\mathcal{J}(C_\lambda)$ equals the complex plane $\mathbb{C}^{\infty}$ for $\|\lambda\| > \lambda^{*}$.
For $0 < \|\lambda\| < \lambda^{**}$, $\mathcal{J}(h_\lambda)$ consists of two rows of Cantor bouquets with crowns facing each other from the right and the left half planes.	For $0 < \|\lambda\| < \lambda^{*}$, $\mathcal{J}(C_\lambda)$ consists of two rows of Cantor bouquets with crowns facing each other from the upper and the lower half planes.
The real parts of the tip of the crowns of Cantor bouquets of $\mathcal{J}(h_\lambda)$, $0 < \|\lambda\| < \lambda^{**}$, in the right half plane (the left half plane) are pushed more towards right (left) as their vertical distances from the real axis increase.	The real parts of the tips of the crowns of Cantor bouquets of $\mathcal{J}(C_\lambda)$, $0 < \|\lambda\| < \lambda^{*}$, have the same imaginary part irrespective of their horizontal distances from the imaginary axis.
For $0 < \|\lambda\| < \lambda^{**}$, $\mathcal{F}(h_\lambda)$ equals the basin of attraction of the real attracting fixed point of $h_\lambda(z)$.	For $0 < \|\lambda\| < \lambda^{*}$, $\mathcal{F}(C_\lambda)$ equals the basin of attraction of the purely imaginary attracting fixed point of $C_\lambda(z)$.

Table 5.1: Comparison between the dynamical properties of $h_\lambda(z) = \frac{\lambda \sinh z}{z}$ and $C_\lambda(z) = \lambda\, i \cos z$.

$h_\lambda(z) = \lambda\dfrac{\sinh z}{z}$, $\lambda > 0$	$f_\lambda(z) = \lambda\dfrac{e^z - 1}{z}$, $\lambda > 0$
$h_\lambda(z)$ is not periodic.	$f_\lambda(z)$ is not periodic.
$h_\lambda(z)$ is an even function.	$f_\lambda(z)$ is neither even nor odd function.
$h_\lambda(z)$ has infinitely many critical values.	$f_\lambda(z)$ has infinitely many critical values.
$h_\lambda(z)$ has only one (finite) asymptotic value, namely 0.	$f_\lambda(z)$ has only one (finite) asymptotic value, namely 0.
$\mathcal{J}(h_\lambda) = Esc(h_\lambda)$ for $\lambda > 0$.	$\mathcal{J}(f_\lambda) = Esc(f_\lambda)$ for $\lambda > 0$.
The bifurcation occurs in the dynamics of $h_\lambda(z)$ at the critical parameter value $\lambda^{**} \approx 1.104$, defined by (5.3.2).	The bifurcation occurs in the dynamics of $f_\lambda(z)$ at the critical parameter value $\lambda^* \approx 0.64761$, defined by (4.3.2).
$\mathcal{J}(h_\lambda)$ contains both the real axis and the imaginary axis for $\lambda > \lambda^{**}$.	$\mathcal{J}(f_\lambda)$ contains only the real axis for $\lambda > \lambda^*$.
For $0 < \lambda < \lambda^{**}$, $\mathcal{J}(h_\lambda)$ consists of two rows of Cantor bouquets with crowns facing each other from the right and the left half planes.	For $0 < \lambda < \lambda^*$, $\mathcal{J}(f_\lambda)$ consists of one row of Cantor bouquets entirely contained in the right half plane.
The real parts of the tips of the crowns of Cantor bouquets of $\mathcal{J}(h_\lambda)$, $0 < \lambda < \lambda^{**}$, in the right half plane (the left half plane) are pushed more towards right (left) as their vertical distances from the real axis increase.	The real parts of the tips of the crowns of Cantor bouquets of $\mathcal{J}(f_\lambda)$, $0 < \lambda < \lambda^*$, are pushed more towards right as their vertical distances from the real axis increase.
For $0 < \lambda < \lambda^{**}$, $\mathcal{F}(h_\lambda)$ equals the complement of the basin of attraction of the real attracting fixed point.	For $0 < \lambda < \lambda^*$, $\mathcal{F}(f_\lambda)$ equals the complement of the basin of attraction of the real attracting fixed point.

Table 5.2: Comparison between the dynamical properties of $h_\lambda = \frac{\lambda \sinh z}{z}$ and $f_\lambda = \frac{\lambda(e^z-1)}{z}$ for $\lambda > 0$.

(a) $\lambda = 1.0 < \lambda^{**}$

(b) $\lambda = 1.2 > \lambda^{**}$

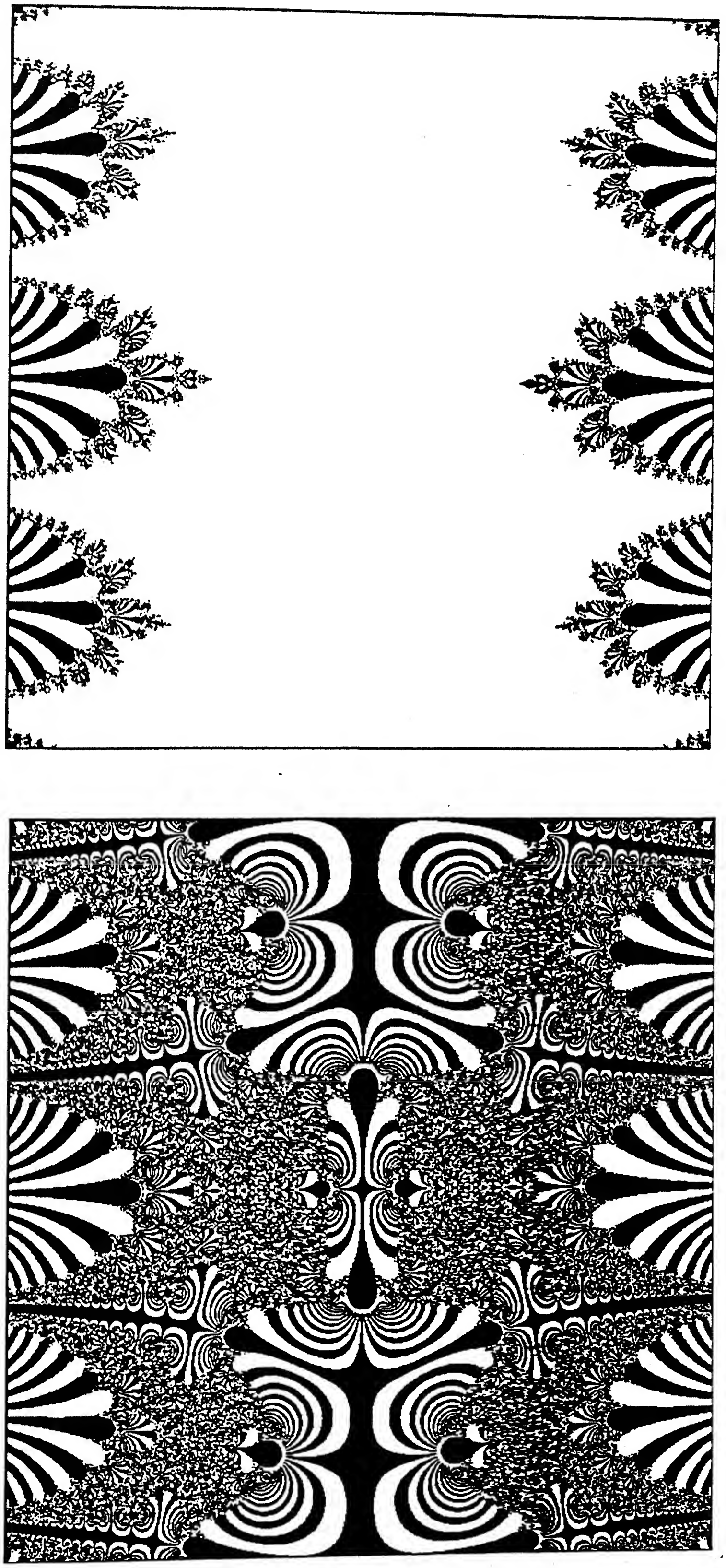


Figure 5.6: Julia sets of $h_\lambda(z)$ for (a) $\lambda = 1.0 < \lambda^{**}$ and (b) $\lambda = 1.2 > \lambda^{**}$.

Chapter 6

Chaotic burst in the dynamics of a class of non-critically finite entire functions

A number of different ways by which dynamical systems become chaotic have been examined in recent years [44, 45, 50]. The period doubling mechanisms is one such way to chaos [45]. Another way yielding a sudden burst into chaos is a collision of an attracting periodic point and a repelling periodic point [50]. Devaney [25, 28] and Devaney and Durkin [33] exhibited the chaotic burst in the dynamics of certain critically finite entire transcendental functions such as $\lambda \exp z$ and $i\lambda \cos z$. While describing the dynamics of $E_\lambda(z) = \lambda \exp z$, $\lambda > 0$, Devaney [28, 33] proved that the Julia set of $E_\lambda(z)$ for $0 < \lambda < (1/e)$ is a nowhere dense subset entirely contained in the right half plane. As soon as the parameter λ crosses the value $(1/e)$, $\mathcal{J}(E_\lambda)$ suddenly explodes and equals to the extended complex plane. This phenomena is referred to as *explosion* in the Julia set or *chaotic burst* in the dynamics of functions in one parameter family $\mathcal{E} \equiv \{E_\lambda(z) = \lambda\ \exp z : \lambda > 0\}$. This type of explosion occurs as well in the family of functions $\mathcal{C} \equiv \{C_\lambda(z) = \lambda\ i \cos z : \lambda \in \mathrm{R}$ and $\lambda \neq 0\}$. The Julia set of $C_\lambda(z)$ for $0 < \lambda < \lambda^* \approx 0.66274$ is a nowhere dense subset of the complex plane while, for $\lambda > \lambda^* \approx 0.66274$, the Julia set $\mathcal{J}(C_\lambda)$ explodes to the extended complex plane [33].

Since the Julia set of a polynomial is compact, this type of chaotic explosion cannot happen in the families of polynomials. However, all the families considered so far to demonstrate explosion in the Julia sets of their functions consist of only critically finite entire functions. In this chapter, a class of non-critically finite entire functions is introduced and it is proved that explosion occurs in the Julia sets of functions in one parameter family generated from each function in this class. Further, the dynamics of functions in one parameter family corresponding to each function in this class is investigated and the occurrence of bifurcation in their dynamics is established.

6.1 The class $\mathcal{G}$ and one parameter family $\mathcal{S}$

Let $\mathcal{F}$ be the class of functions defined by

$$\mathcal{F} = \left\{ f(z) : \begin{array}{ll} (i) & f(z) \text{ is an entire function having order } \rho \text{ with } (1/2) \le \rho < 1 \\ (ii) & f(z) \text{ has only negative real zeros in the complex plane} \\ (iii) & |f(-x)| \le f(0) = 1 \text{ for all } x > 0 \\ (iv) & \lim_{x\to\infty} f(-x) = 0 \end{array} \right\} \qquad (6.1.1)$$

and $\mathcal{G}$ be the class of functions defined by

$$\mathcal{G} = \left\{ g(z) = f(z^2) : f \in \mathcal{F} \right\} .$$

For a function $g \in \mathcal{G}$, let

$$\mathcal{S} \equiv \{ g_\lambda(z) = \lambda g(z) : g \in \mathcal{G} \text{ and } \lambda \in \mathbb{R} \setminus \{0\} \}$$

be one parameter family of entire transcendental functions. In the present chapter, the dynamics of $g_\lambda \in \mathcal{S}$ is studied. Some of the basic properties of the functions $g \in \mathcal{G}$ that are needed in the sequel are developed in the present section. In Section 6.2, the dynamics of functions $g_\lambda \in \mathcal{S}$ on the real line $\mathbb{R}$ is investigated. It is shown that there exists a critical parameter value $\lambda_g^* > 0$ such that bifurcation in the dynamics of functions in $\mathcal{S}$ for $x \in \mathbb{R}$ occurs at $|\lambda| = \lambda_g^*$. In Section 6.3, the dynamics of $g_\lambda \in \mathcal{S}$ for $z \in \mathbb{C}$ is described and the

chaotic burst in the dynamics of functions in one parameter family $\mathcal{S}$ is exhibited. It is proved that the Fatou set of $g_\lambda(z)$ is the basin of attraction of the attracting real fixed point when $0 < |\lambda| < \lambda_g^*$, is the parabolic domain corresponding to the rationally indifferent real fixed point when $|\lambda| = \lambda_g^*$ and is an empty set when $|\lambda| > \lambda_g^*$. Further, in this section, the characterization of the Julia set of $g_\lambda(z)$, $\lambda \in \mathbb{R} \setminus \{0\}$, as the closure of the set of all escaping points is obtained. In Section 6.4, certain interesting examples of the family $\mathcal{S}$, viz., (i) $\mathcal{I} = \{\lambda I_0(z) \ : \ \lambda \in \mathbb{R}\setminus\{0\}\}$, where $I_0 \in \mathcal{G}$ is the well known modified Bessel function of zero order arising as the denominator of the separately convergent modified general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\dfrac{-z^2/(2n)^2}{1+z^2/(2n)^2}\right)$ and (ii) $\mathcal{M}_k = \{\lambda G_{2k}(z) \ : \ G_{2k}(z) = F_{2k}(iz)/F_{2k}(0), \ \lambda \in \mathbb{R}\setminus\{0\}\}$, where $G_{2k} \in \mathcal{G}$ with fixed $k = 1, 2, \ldots$ and $F_{2k}(z) = \int_0^\infty e^{-t^{2k}} \cos zt \, dt$ are given. Finally, the pictures of the Julia sets of functions in the family $\mathcal{I}$ are computationally generated for various values of λ.

First, we develop some of the basic properties of the functions in the class $\mathcal{G}$. The following proposition shows that all the functions $g \in \mathcal{G}$ are non-critically finite.

Proposition 6.1.1. *Let $g \in \mathcal{G}$. Then, $g(z)$ possesses infinitely many real critical values.*

Proof. Since $g \in \mathcal{G}$, there exist a function $f \in \mathcal{F}$ such that $g(z) = f(z^2)$. Since, by (6.1.1*(i)*), $f(z)$ has non-integral order, it has infinitely many distinct zeros ([100] p252). Again, by (6.1.1*(i)*), (6.1.1*(ii)*) and ([100], p266), the zeros of $f'(z)$ separate the zeros of $f(z)$. Therefore, the function $f'(z)$ has infinitely many negative real zeros and consequently $g'(z)$ has infinitely many purely imaginary zeros. This proves that the function $g(z)$ has infinitely many purely imaginary critical points w_n, $n = 1, 2, \cdots$. Obviously, the point $w_0 = 0$ is also a critical point of $g(z)$. Let $\{w_n\}_{n=0}^\infty$ be the sequence of purely imaginary numbers such that $g'(w_n) = 0$ for $n = 0, 1, \cdots$, $w_0 = 0$ and $w_n \to \infty$ as $n \to \infty$. Since $g'(-z) = -g'(z)$, $g'(z)$ becomes zero at the points $w_{-n} = -w_n$, for $n = 1, 2, \cdots$. Since the zeros of $f'(z)$ and $f(z)$ separate each other so are the zeros of $g'(z)$ and $g(z)$. Therefore, $g(w_n) \neq 0$ for all n. Since by (6.1.1*(iv)*), $\lim_{y\to\pm\infty} g(iy) = \lim_{y\to\infty} f(-y^2) = 0$,

$\lim_{n\to\pm\infty} g(w_n) = 0$. In view of $g(w_n) \neq 0$ for all n, it follows that $g(z)$ has infinitely many critical values. Further, $g(iy) = f(-y^2) \in \mathbb{R}$, for all $y \in \mathbb{R}$. Therefore, $g(w_n) \in \mathbb{R}$ for all n, proving that $g(z)$ has only real critical values. □

In the following proposition, it is proved that the functions in the class $\mathcal{G}$ have only one (finite) asymptotic value:

Proposition 6.1.2. *Let $g \in \mathcal{G}$. Then, $w = 0$ is the only finite asymptotic value of $g(z)$.*

Proof. We first observe that the functions $g(z) = f(z^2)$ and $f(z)$ have the same asymptotic values. Since, by (6.1.1*(i)*), the order of $f(z)$ is less than one, by Denjoy-Ahlfors Theorem (c.f. Theorem 1.3.10), it has atmost one finite asymptotic value. By (6.1.1*(iv)*), the point $w = 0$ is the asymptotic value of $f(z)$. Therefore, it follows that the function $f(z)$ and consequently the function $g(z) = f(z^2)$ has only one asymptotic value, namely 0. □

6.2 Bifurcation in the dynamics of functions in $\mathcal{S}$

In this section, the dynamics of $g_\lambda \in \mathcal{S}$ on the real line is investigated. For this purpose, the existence and the nature of fixed points on the real line are examined in Theorem 6.2.1. The dynamics of $g_\lambda(x)$ for $x \in \mathbb{R}$ is described in Theorem 6.2.2. It follows from Theorem 6.2.2 that there exists a parameter value $\lambda_g^* > 0$ such that bifurcation in the dynamics of $g_\lambda(x)$, $x \in \mathbb{R}$, occurs at $|\lambda| = \lambda_g^*$.

Since by (6.1.1*(i)*) and (6.1.1*(ii)*), $f(z)$ is an entire function with only negative real zeros and has order less than one, it admits a Hadamard factorization ([100], p271) of the form:

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right), \quad z_n > 0 . \tag{6.2.1}$$

Therefore,

$$g(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{z_n}\right), \quad z_n > 0 . \tag{6.2.2}$$

Set. $g(x) = \prod_{n=1}^{\infty}\left(1 + \frac{x^2}{z_n}\right)$ for $x \in$ R. It is easily seen that the function $g(z)$ is even and the coefficients of Taylors series of $g(z)$ are non-negative. Hence, the function $g(x)$ and $g'(x)$ are strictly increasing positive valued functions for $x > 0$. Therefore, the function $\phi(x) = g(x) - xg'(x)$, is strictly decreasing in the interval $[0, \infty)$. Since $g(x)$ is an even function, the function $\phi(x)$ is also an even function. Consequently, since $\phi(0) = 1$ and $\phi(x_0) < 0$ for sufficiently large values of x_0, there exists a unique $x^* \equiv x^*(g)$ in the interval $(0, x_0)$ such that

$$\phi(x)\begin{cases} > 0 & \text{for } |x| < x^* \\ = 0 & \text{for } |x| = x^* \\ < 0 & \text{for } |x| > x^* \end{cases} \tag{6.2.3}$$

Throughout in the sequel, we denote

$$\lambda_g^* = \frac{1}{g'(\ x^*\ )} \tag{6.2.4}$$

where, x^* is the unique positive real root of the equation $\phi(x) = 0$.

Remark 6.2.1. *The root x^* of the equation $\phi(x) = 0$ can be computed using the product form expression (6.2.2) of $g(z)$ as follows: For $x > 0$, by taking logarithmic derivative of $g(x)$ it follows that $\frac{g'(x)}{g(x)} = \sum_{n=1}^{\infty}\frac{2x}{z_n+x^2}$. Since $\phi(x) = g(x)\left(1 - \frac{g'(x)}{g(x)}\right)$ and $g(x) > 0$ for all $x > 0$, x^* is the root of the equation $\phi(x) = 0$ if and only if x^* is the root of the equation $1 - \frac{g'(x)}{g(x)} = 0$. Therefore, x^* is the root of the equation*

$$\sum_{n=1}^{\infty}\frac{2x^2}{z_n + x^2} = 1\,.$$

Now, the root of the equation $\sum_{n=1}^{\infty}\frac{2x^2}{z_n+x^2} = 1$ can be easily computed by using simple numerical methods.

The following theorem describes the nature of fixed points of $g_\lambda(x)$ for $x \in$ R.

Theorem 6.2.1. *Let $g_\lambda(x) = \lambda g(x)$ for $x \in$ R, where λ is a non-zero real parameter.*

(a) *If $0 < \lambda < \lambda_g^*$, $g_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ (say) with $0 < a_\lambda < r_\lambda$.*

(b) *If* $\lambda = \lambda_g^*$, $g_\lambda(x)$ *has a unique rationally indifferent fixed point at* $x = x^*$ *, where* x^* *is given by (6.2.4).*

(c) *If* $\lambda > \lambda_g^*$, $g_\lambda(x)$ *has no fixed points.*

(d) *If* $-\lambda_g^* < \lambda < 0$, $g_\lambda(x)$ *has an attracting fixed point* a_λ *and a repelling fixed point* r_λ *(say) with* $r_\lambda < a_\lambda < 0$.

(e) *If* $\lambda = -\lambda_g^*$, $g_\lambda(x)$ *has a unique rationally indifferent fixed point at* $x = -x^*$ *, where* x^* *is given by (6.2.4).*

(f) *If* $\lambda < -\lambda_g^*$, $g_\lambda(x)$ *has no fixed points.*

Proof. Define $\Psi_\lambda(x) = g_\lambda(x) - x = \lambda g(x) - x$. The zeros of $\Psi_\lambda(x)$ are fixed points of $g_\lambda(x)$. It is easily seen from (6.2.2) that the function $g(z)$ has a Taylors series expansion at $z = 0$ of the form:

$$g(z) = 1 + \sum_{n=1}^{\infty} a_n z^{2n}, \qquad a_n > 0\,.$$

Obviously, $g(z)$ and $g''(z)$ are even functions and $g'(z)$ is an odd function. Therefore,

(i) $\Psi_\lambda(x)$ is continuously differentiable in R . For $|x|$ sufficiently large, $\Psi_\lambda(x)$ is positive if $\lambda > 0$, and is negative if $\lambda < 0$.

(ii) If $\lambda > 0$, there exists a unique real number $\tilde{x} \equiv \tilde{x}(\lambda) > 0$ such that $\Psi'_\lambda(x) < 0$ for $x < \tilde{x}$, $\Psi'_\lambda(x) > 0$ for $x > \tilde{x}$ and $\Psi'_\lambda(\tilde{x}) = 0$. Similarly, if $\lambda < 0$, there exists a unique real number $\tilde{x} \equiv \tilde{x}(\lambda) < 0$ such that $\Psi'_\lambda(x) > 0$ for $x < \tilde{x}$, $\Psi'_\lambda(x) < 0$ for $x > \tilde{x}$ and $\Psi'_\lambda(\tilde{x}) = 0$. Further, $\Psi'_\lambda(x)$ is strictly increasing for $\lambda > 0$ and, strictly decreasing for $\lambda < 0$.

(iii) The function $\Psi_\lambda(x)$ attains its unique local minimum at $x = \tilde{x}$ for $\lambda > 0$ and unique local maximum at $x = \tilde{x}$ for $\lambda < 0$.

Now, the rest of the proof is similar to that of Theorem 5.3.1. □

The following theorem describes the dynamics of $g_\lambda \in \mathcal{S}$ on the real line R .

Theorem 6.2.2. *Let* $g_\lambda\ (x) = \lambda g(x)$ *for* $x \in$ R *, where* λ *is a non-zero real parameter.*

(a) *If* $0 < \lambda < \lambda_g^*$, $g_\lambda^n(x) \to a_\lambda$ *as* $n \to \infty$ *for* $|x| < r_\lambda$ *and* $g_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $|x| > r_\lambda$, *where* a_λ *and* r_λ *are the attracting and the repelling fixed points of* $g_\lambda(x)$ *respectively.*

(b) *If* $\lambda = \lambda_g^*$, $g_\lambda^n(x) \to x^*$ *as* $n \to \infty$ *for* $|x| < x^*$ *and* $g_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $|x| > x^*$, *where* x^* *is the rationally indifferent fixed point of* $g_\lambda(x)$.

(c) *If* $\lambda > \lambda_g^*$, $g_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for all* $x \in \mathrm{R}$.

(d) *If* $-\lambda_g^* < \lambda < 0$, $g_\lambda^n(x) \to a_\lambda$ *as* $n \to \infty$ *for* $|x| < -r_\lambda$ *and* $g_\lambda^n(x) \to -\infty$ *as* $n \to \infty$ *for* $|x| > -r_\lambda$, *where* a_λ *and* r_λ *are the attracting and the repelling fixed points of* $g_\lambda(x)$ *respectively.*

(e) *If* $\lambda = -\lambda_g^*$, $g_\lambda^n(x) \to -x^*$ *as* $n \to \infty$ *for* $|x| < x^*$ *and* $g_\lambda^n(x) \to -\infty$ *as* $n \to \infty$ *for* $|x| > x^*$, *where* $-x^*$ *is the rationally indifferent fixed point of* $g_\lambda(x)$ *and* x^* *is given by (6.2.4).*

(f) *If* $\lambda < -\lambda_g^*$, $g_\lambda^n(x) \to -\infty$ *as* $n \to \infty$ *for all* $x \in \mathrm{R}$.

Proof. The proof follows on the lines similar to that of Theorem 5.3.2 and hence, is omitted.

□

Remark 6.2.2. *Theorem 6.2.2 shows that the bifurcation occurs in the dynamics of the functions* $g_\lambda(x)$, $x \in \mathrm{R}$, *in the family* $\mathcal{S}$. *To see this, let*

$$\hat{x_\lambda} = \begin{cases} |\, r_\lambda \,| & \text{if } 0 < |\lambda| < \lambda_g^* \\ x^* & \text{if } |\lambda| = \lambda_g^* \end{cases}$$

where, r_λ *is the repelling real fixed point of* $g_\lambda(z)$ *for* $0 < |\lambda| < \lambda_g^*$ *and* x^* *is given by (6.2.4). If* $0 < |\lambda| \le \lambda_g^*$, *it follows from Theorem 6.2.2 that under iteration of* g_λ *the orbits of all the points in absolute value less than* $\hat{x_\lambda}$ *remain bounded and the orbits of all the points in absolute value greater than* $\hat{x_\lambda}$ *become unbounded; while, if* $|\lambda| > \lambda_g^*$, *there is no real point whose orbit remain bounded under iteration of* f_λ. *Thus, bifurcation in the dynamics of* $g_\lambda(x)$, $x \in \mathrm{R}$, *occurs at the parameter value* $|\lambda| = \lambda_g^*$. *Bifurcation diagram for the dynamics of* $g_\lambda(x)$ *is similar to that of Figure 5.3.*

6.3 Chaotic burst in the dynamics of functions in $\mathcal{S}$

The chaotic burst in the dynamics of functions in one parameter family $\mathcal{S}$ is exhibited by describing the dynamics of $g_\lambda \in \mathcal{S}$ for $z \in \mathbb{C}$ in the present section. Three different cases, viz. $0 < |\lambda| < \lambda_g^*$, $|\lambda| = \lambda_g^*$ and $|\lambda| > \lambda_g^*$ are considered in Subsections 6.3.I, 6.3.II and 6.3.III respectively.

6.3.I Dynamics of $g_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < |\lambda| < \lambda_g^*$

The dynamics $g_\lambda \in \mathcal{S}$ for $z \in \mathbb{C}$ and $0 < |\lambda| < \lambda_g^*$, where λ_g^* is defined by (6.2.4) is investigated in this subsection. In this case, it is proved in Theorem 6.3.1 that the Fatou set of $g_\lambda(z)$ is the basin of attraction of the attracting real fixed point of $g_\lambda(z)$. The computationally useful characterization of the Julia set of $g_\lambda(z)$ in this case is found in Theorem 6.3.2.

If $0 < |\lambda| < \lambda_g^*$, by Theorem 6.2.1(a)and (d) it follows that $g_\lambda(z)$ has a real attractive fixed point a_λ and a real repelling fixed point r_λ such that $\tilde{x}$ with $g'_\lambda(\tilde{x}) = 1$ satisfies $0 < |a_\lambda| < |\tilde{x}| < |r_\lambda|$. The basin of attraction $\mathsf{A}(a_\lambda)$ of the attractive fixed point a_λ is defined as

$$\mathsf{A}(a_\lambda) = \{z \in \mathbb{C} : g_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}.$$

First, in the following proposition it is shown that the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point of $g_\lambda(z)$ for $0 < |\lambda| < \lambda_g^*$ contains the set of all points in the complex plane which get mapped inside the disk $D_{|\tilde{x}|}(0)$ centered at 0 and having radius $|\tilde{x}|$, where $\tilde{x}$ is the unique real root of the equation $g'_\lambda(x) = 1$:

Proposition 6.3.1. *Let $g_\lambda \in \mathcal{S}$ and $0 < |\lambda| < \lambda_g^*$. Then, the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point a_λ contains the set $D = \{z \in \mathbb{C} : |g_\lambda(z)| < |\tilde{x}|\}$, where $\tilde{x}$ is the real number such that $g'_\lambda(\tilde{x}) = 1$. Further, $|\tilde{x}| > x^*$, where x^* is given by (6.2.4).*

Proof. By (6.2.2), the coefficients of Taylors series of $g(z)$ are non-negative and $g(z)$ is even.

Therefore, the function $g'(x)$ is strictly increasing odd function for $x \in \mathbb{R}$. If $0 < \lambda < \lambda^*$, $\frac{1}{g'(\tilde{x})} < \frac{1}{g'(x^*)}$. Consequently, $\tilde{x} > x^*$. If $-\lambda_g^* < \lambda < 0$ then $-\lambda_g^* = \frac{-1}{g'(x^*)} = \frac{1}{g'(-x^*)} < \lambda = \frac{1}{g'(\tilde{x})}$. Since $g'(x)$ is strictly increasing function it follows that $\tilde{x} < -x^* < 0$. Thus, $|\tilde{x}| > x^*$.

Adopting the lines of proof of Proposition 3.3.1 and using the equation (6.2.3) instead of (3.3.1), it follows that for $0 < |\lambda| < \lambda_g^*$, $g_\lambda(z)$ maps the open disk $D_{|\tilde{x}|}(0)$ centered at origin and having radius $|\tilde{x}|$ into itself. Since by (6.1.1*(ii)*), $f(z)$ has only negative real zeros, the function $g_\lambda(z)$ has only purely imaginary zeros. By (6.1.1*(iii)*) and $g(z) = f(z^2)$, g_λ at purely imaginary points satisfies the inequality $|g_\lambda(ix)| \leq |g_\lambda(0)| = |\lambda|$ for all $x \in \mathbb{R}$. Therfore, the curve $\gamma = \{z \in \mathbb{C} : \Re(z) > 0$ and $|g_\lambda(z)| = |\tilde{x}|\}$ is connected and not self intersecting ([74], vol.1, p376). So, $D = \{z \in \mathbb{C} : |g_\lambda(z)| < |\tilde{x}|\}$ is a simply connected domain. Since $g_\lambda(z)$ maps D into $D_{|\tilde{x}|}(0)$ and $g_\lambda(D_{|\tilde{x}|}(0)) \subseteq D_{|\tilde{x}|}(0)$, by Schwarz lemma ([30], p264), $g_\lambda^n(z) \to a_\lambda$ as $n \to \infty$ for all $z \in D$. Thus, $\mathsf{A}(a_\lambda) \supset D$. □

Remark 6.3.1. *(i) By (6.1.1(iii)) and $g(z) = f(z^2)$, it follows that $|g_\lambda(ix)| \leq |\lambda|$ for $x \in \mathbb{R}$. Therefore, by Proposition 6.3.1, the imaginary axis is contained in the basin of attraction $\mathsf{A}(a_\lambda)$ of the real attracting fixed point of $g_\lambda(z)$ for $0 < |\lambda| < \lambda_g^*$. Consequently, the basin of attraction $\mathsf{A}(a_\lambda)$ is an unbounded set.*

(ii) By Theorem 6.2.2(a) and (d), $g_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $|x| < |r_\lambda|$ and hence, $\mathsf{A}(a_\lambda)$ also contains the interval $(-|r_\lambda|, |r_\lambda|)$ in the real axis.

The following theorem describes the Fatou set of $g_\lambda(z)$ for $0 < |\lambda| < \lambda_g^*$.

Theorem 6.3.1. *Let $g_\lambda \in S$ and $\mathsf{A}(a_\lambda)$ be the basin of attraction of the real attracting fixed point a_λ of $g_\lambda(z)$ for $0 < |\lambda| < \lambda_g^*$. Then, for $0 < |\lambda| < \lambda_g^*$, $\mathcal{F}(g_\lambda) = \mathsf{A}(a_\lambda)$.*

Proof. Since $g(z) = f(z^2)$, the zeros of $f'(z^2)$ are the zeros of $g'(z)$. Further, $g'(0) = 0$. By (6.1.1*(i)*), (6.1.1*(ii)*) and ([100], p266), the zeros of $f'(z)$ separate the zeros of $f(z)$. Therefore, the function $f'(z)$ has infinitely many negative real zeros only and consequently

$g'(z)$ has only purely imaginary zeros. By (6.1.1*(iii)*) and $g(z) = f(z^2)$, $|g_\lambda(ix)| \leq |\lambda|$ for $x \in$ R follows. This shows that the absolute value of all the critical values of $g_\lambda(z)$ are less than are equal to $|\lambda|$. By Proposition 6.1.2, the function $g_\lambda(z)$ has only one finite asymptotic value, namely 0. Therefore, the absolute value of all the singular values of $g_\lambda(z)$ are less than are equal to $|\lambda|$. Since $|\lambda| < |\tilde{x}|$ for $0 < |\lambda| < \lambda_g^*$, by Proposition 6.3.1 it follows that all the singular values of $g_\lambda(z)$ lie in the attractive basin $A(a_\lambda)$. Since $A(a_\lambda)$ is invariant, the forward orbits of all singular values also lie in $A(a_\lambda)$.

Further, the Fatou set of $g_\lambda(z)$ has no attractive basin other than $A(a_\lambda)$. To see this, assume, if possible, $A(z_\lambda)$ is an attractive basin of the attracting periodic point $z_\lambda \neq a_\lambda$. Obviously, $A(z_\lambda) \cap A(a_\lambda) = \emptyset$. Then, by Theorem 1.1.8, $A(z_\lambda)$ contains atleast one critical point, and hence atleast one critical value and its orbit. This contradicts the fact that all singular values and its forward orbits are contained in $A(a_\lambda)$, since $A(z_\lambda) \cap A(a_\lambda) = \emptyset$ for $z_\lambda \neq a_\lambda$.

The Fatou set of $g_\lambda(z)$ cannot contain a parabolic domain. For, if the Fatou set of $g_\lambda(z)$ contains a parabolic domain U, by Theorem 1.1.8, the invariant domain U must contain atleast one critical point and hence atleast one critical value and its orbit, which leads to a contradiction that all the singular values and its forward orbits lie in $A(a_\lambda)$.

Again, the Fatou set of $g_\lambda(z)$ cannot contain a Siegel disk. For, if possible, the Fatou set of $g_\lambda(z)$ contains a Siegel disk U, by Theorem 1.1.9, the boundary of the Siegel disk U is contained in the forward orbits of all the singular values of $g_\lambda(z)$. But, all the singular values and there forward orbits are contained in $A(a_\lambda)$, giving a contradiction.

Since all the singular values of $g_\lambda(z)$ form a bounded set, there does not exist a component U of the Fatou set of $g_\lambda(z)$ such that $g_\lambda^n|_U \to \infty$ as $n \to \infty$ [40]. Therefore, the Fatou set of $g_\lambda(z)$ has no Baker domain (Domain at infinity).

Finally, the Fatou set of $g_\lambda(z)$ does not contain a wandering domain. To see this, let the Fatou set of $g_\lambda(z)$ contains a wandering domain. Let U be one of the components of

the wandering domain and $w \in U$. Since $\{g_\lambda^n(z)\}$ is normal in some neighborhood $N(w)$ of w, the limit function of $\{g_\lambda^n(z)|_{N(w)}\}$ is finite and the finite limit function of $\{g_\lambda^n(z)|_{N(w)}\}$ is contained in the forward orbits of all singular values of $g_\lambda(z)$ ([20], c.f. 1.1.10). This gives a contradiction, since all the singular values and their forward orbits are contained in the attractive basin $\mathsf{A}(a_\lambda)$.

Thus, all the possible stable domains other than the basin of attraction $\mathsf{A}(a_\lambda)$ of the Fatou set are excluded. Therefore, the Fatou set of $g_\lambda(z)$ equals $\mathsf{A}(a_\lambda)$. □

Remark 6.3.2. *(i) If $0 < |\lambda| < \lambda_g^*$, Theorem 6.3.1 provides a characterization of the Julia set of $g_\lambda(z)$ as the complement of the basin of attraction $\mathsf{A}(a_\lambda)$ of the attracting real fixed point.*

(ii) Since the Fatou set is completely invariant, the basin of attraction $\mathsf{A}(a_\lambda)$ of the attracting real fixed point of $g_\lambda(z)$, $0 < |\lambda| < \lambda_g^*$, is completely invariant.

The following theorem characterizes the Julia set of $g_\lambda(z)$ for $0 < |\lambda| < \lambda_g^*$ as the closure of the set of escaping points:

Theorem 6.3.2. *Let $g_\lambda \in \mathcal{S}$ and $Esc(g_\lambda) = clo\ \{z \in \mathbb{C} : g_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$ be the closure of the set of escaping points of $g_\lambda(z)$. If $0 < |\lambda| < \lambda_g^*$, then the Julia set $\mathcal{J}(g_\lambda) = Esc(g_\lambda)$.*

Proof. By using Theorem 6.2.2(a) and (d) instead of Theorem 5.3.2(a) and (d), the proof of the inclusion relation $\mathcal{J}(g_\lambda) \subseteq \mathrm{Esc}(g_\lambda)$ follows on the lines of proof of Theorem 5.4.1.

To prove the reverse inclusion $\mathrm{Esc}(g_\lambda) \subseteq \mathcal{J}(g_\lambda)$, we note that if $z_0 \in \mathrm{Esc}(g_\lambda)$ then $z_0 \notin \mathsf{A}(a_\lambda)$. Consequently, by Theorem 6.3.1, $z_0 \in \mathcal{J}(g_\lambda)$ and $\mathrm{Esc}(g_\lambda) \subseteq \mathcal{J}(g_\lambda)$ follows. □

6.3.II Dynamics of $g_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| = \lambda_g^*$

In this subsection, the dynamics $g_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| = \lambda_g^*$ is described, where λ_g^* is defined by (6.2.4). Theorem 6.3.3 shows that the Fatou set of $g_\lambda(z)$ is the parabolic domain corresponding to the rationally indifferent real fixed point of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$. Theorem 6.3.4 gives the computationally useful characterization of the Julia set of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$.

If $\lambda = \lambda_g^*$, by Theorem 6.2.1(b), the function $g_\lambda(z)$ has a unique rationally indifferent real fixed point at $x = x^*$, where x^* is given by (6.2.4). Let

$$\mathsf{U}(x^*) = \{z \in \mathbb{C} : g_\lambda^n(z) \to x^* \text{ as } n \to \infty\}$$

be the parabolic domain corresponding to the indifferent fixed point x^* of $g_\lambda(z)$ for $\lambda = \lambda_g^*$. If $\lambda = -\lambda_g^*$, by Theorem 6.2.1(d), the function $g_\lambda(z)$ has a unique real indifferent fixed point at $x = -x^*$, where x^* is given by (6.2.4). Let $\mathsf{U}(-x^*)$ be the parabolic domain corresponding to the indifferent fixed point $-x^*$ of $g_\lambda(z)$ for $\lambda = -\lambda_g^*$ *i.e.*,

$$\mathsf{U}(-x^*) = \{z \in \mathbb{C} : g_\lambda^n(z) \to -x^* \text{ as } n \to \infty\}$$

First, we prove in the following proposition that the parabolic domain $\mathsf{U}(x^*)$ or $\mathsf{U}(-x^*)$ corresponding to the real indifferent fixed point of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$ contains the set of all points in the complex plane which get mapped inside the disk $D_{x^*}(0)$ centered at 0 and having radius x^*, where x^* is given by (6.2.4).

Proposition 6.3.2. *Let $g_\lambda \in \mathcal{S}$ and $|\lambda| = \lambda_g^*$. Then, the parabolic domain $\mathsf{U}(x^*)$ (or $\mathsf{U}(-x^*)$) corresponding to the unique indifferent real fixed points x^* (or $-x^*$) of $g_\lambda(z)$ contain $D = \{z \in \mathbb{C} : |g_\lambda(z)| < x^*\}$, where x^* is the real number such that $g_\lambda'(x^*) = 1$.*

Proof. Let $g_\lambda(z) = \lambda g(z)$ for $z \in \mathbb{C}$, where $g(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{z_n}\right)$ and $z_n > 0$. We show that $g_\lambda(z)$ maps the open disk $D_{x^*}(0)$, centered at origin and having radius x^* into

itself.

Case I ($\lambda = \lambda_g^*$):

Since, by (6.2.3), $g'(x^*)\left(\frac{g(x^*)}{g'(x^*)} - x^*\right) = g(x^*) - x^* g'(x^*) = \phi(x^*) = 0$. Consequently, $\frac{g(x^*)}{g'(x^*)} = x^*$, since $g'(x^*) > 0$. Therefore, $g_\lambda(x) = \lambda g(x) = \frac{1}{g'(x^*)} g(x) < \frac{g(x^*)}{g'(x^*)} = x^*$ for $x < x^*$. By (6.2.2), the coefficients of Taylors series of $g(z)$ are non-negative and hence $|g_\lambda(z)| \leq g_\lambda(|z|)$ follows. Now, $\max_{|z|=x^*} |g_\lambda(z)| \leq g_\lambda(x^*) = \frac{g(x^*)}{g'(x^*)} = x^*$. The maximum modulus principle gives $|g_\lambda(z)| < x^*$ for $z \in D_{x^*}(0)$. Thus, $g_\lambda(z)$ maps the open disk $D_{x^*}(0)$ into itself.

Case II ($\lambda = -\lambda_g^*$):

Since, by (6.2.3) and the fact that $g'(x)$ is strictly increasing,

$$g'(-x^*)\left(\frac{g(-x^*)}{g'(-x^*)} - (-x^*)\right) = g(-x^*) + x^* g'(-x^*) = \phi(-x^*) = 0\,,$$

it follows that $0 > \frac{g(x^*)}{g'(x^*)} = -x^*$. Consequently, $|g_\lambda(x)| = |\lambda| g(x) = \left|\frac{1}{g'(-x^*)}\right| g(x) < \left|\frac{g(x^*)}{g'(-x^*)}\right| = x^*$ for $|x| < x^*$. Therefore, $\max_{|z|=x^*} |g_\lambda(z)| \leq |g_\lambda(-x^*)| \leq |\frac{g(x^*)}{g'(x^*)}| = x^*$, since $|g_\lambda(z)| \leq g_\lambda(|z|)$. Now, as in Case I, $|g_\lambda(z)| < x^*$ for $z \in D_{x^*}(0)$. Thus, $g_\lambda(z)$ maps the open disk $D_{x^*}(0)$ into itself.

The rest of the proof is similar to that of Proposition 6.3.1. □

Remark 6.3.3. *(i) By (6.1.1(iii)) and $g(z) = f(z^2)$, it follows that $|g_\lambda(ix)| \leq |\lambda|$ for $x \in \mathrm{R}$. Therefore, by Proposition 6.3.2, the imaginary axis is contained in the parabolic domain corresponding to the rationally indifferent real fixed point of $g_\lambda(z)$ for $\lambda = \lambda_g^*$. Consequently, the parabolic domain corresponding to the rationally indifferent real fixed point is an unbounded set.*

(ii) By Theorem 6.2.2(b), if $\lambda = \lambda_g^*$, $g_\lambda^n(x) \to x^*$ as $n \to \infty$ for $|x| < x^*$ and $g_\lambda^n(x) \to \infty$ as $n \to \infty$ for $|x| > x^*$. Therefore, $\mathrm{U}(x^*)$ contains the interval $(-x^*, x^*)$. Similarly, by Theorem 6.2.2(e), if $\lambda = -\lambda_g^*$, $g_\lambda^n(x) \to -x^*$ as $n \to \infty$ for $|x| < x^*$ and hence $\mathrm{U}(-x^*)$ also contains the interval $(-x^*, x^*)$.

The following theorem shows that for $|\lambda| = \lambda_g^*$, the Fatou set of $g_\lambda(z)$ is the parabolic domain corresponding to the rationally indifferent real fixed point.

Theorem 6.3.3. *Let $g_\lambda \in \mathcal{S}$ and $|\lambda| = \lambda_g^*$. Then, $\mathcal{F}(g_\lambda) = \mathsf{U}$, where $\mathsf{U} \equiv \mathsf{U}(x^*)$ or $\mathsf{U}(-x^*)$, is the parabolic domain corresponding to the unique indifferent real fixed point of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$.*

Proof. By Proposition 6.3.2, $D = \{z : |g_\lambda(z)| < x^*\} \subseteq \mathsf{U}$. Again, as in the proof of Proposition 6.3.2, $g_\lambda(D_{x^*}(0)) \subseteq D_{x^*}(0)$ so that $D_{x^*}(0) \subseteq D$. Now, following the proof of Theorem 6.3.1, all the singular values of $g_\lambda(z)$ are in absolute value less than are equal to λ_g^* in this case. Since $g_{\lambda_g^*}(0) = \lambda_g^* \in \mathsf{U}$ and U is a parabolic domain, $|\lambda| < |x^*|$. Therefore, $\mathsf{U} \supseteq D_{x^*}(0) \supseteq D_{\lambda_g^*}(0)$ and consequently, all the singular values of $g_\lambda(z)$ lie in the parabolic domain U. Since U is invariant, the forward orbits of all singular values also lie in U.

The Fatou set of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$ has no parabolic domain other than $\mathsf{U}(x^*)$. For, if possible, let $\mathsf{U}(z_1)$ be another parabolic domain whose boundary contains a periodic point z_1, where z_1 is different from x^*. Obviously, $\mathsf{U}(z_1) \cap \mathsf{U}(x^*) = \emptyset$. By Theorem 1.1.8, $\mathsf{U}(z_1)$ contains atleast one critical point, and hence atleast one critical value and its orbit. This contradicts the fact that all the singular values lie inside $\mathsf{U}(x^*)$. Similarly, if $\lambda = -\lambda_g^*$, the Fatou set of $g_\lambda(z)$ has no parabolic domain other than $\mathsf{U}(-x^*)$.

Further, the Fatou set of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$ has no attractive basin. For, if the Fatou set of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$ contains an attractive basin $\mathsf{A}(z_0)$ of an attracting periodic point z_0, by Theorem 1.1.8, the attractive basin $\mathsf{A}(z_0)$ contains atleast one critical point and hence atleast one critical value and its orbit, which leads to a contradiction.

The proofs of the fact that the Fatou set of $g_\lambda(z)$ for $\lambda = \lambda_g^*$ does not contain Siegel disk, Baker domain and wandering domain are similar to that of Theorem 6.3.1. Thus, all the possible stable domains other than the parabolic domain $\mathsf{U}(x^*)$ of $g_\lambda(z)$ of the Fatou set are excluded and the Fatou set of $g_\lambda(z)$ for $\lambda = \lambda_g^*$ equals the parabolic domain $\mathsf{U}(x^*)$. Similarly, for $\lambda = -\lambda_g^*$, the Fatou set of $g_\lambda(z)$ equals the parabolic domain $\mathsf{U}(-x^*)$ $\square$

Remark 6.3.4. *Since the Fatou set is completely invariant, it follows that the parabolic domain* U *in Theorem* 6.3.3, *is completely invariant.*

The following theorem gives computationally useful characterization of the Julia set $\mathcal{J}(g_\lambda)$ as the closure of the set of escaping points of $g_\lambda(z)$ for $|\lambda| = \lambda_g^*$.

Theorem 6.3.4. *Let* $g_\lambda \in \mathcal{S}$ *and* $Esc(g_\lambda) = clo\ \{z \in \mathbb{C} : g_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$ *be the closure of the set of escaping points of* $g_\lambda(z)$. *If* $|\lambda| = \lambda_g^*$, *then the Julia set* $\mathcal{J}(g_\lambda) = Esc(g_\lambda)$.

Proof. By using Theorem 6.2.2(b) and (e) instead of Theorem 5.3.2(a) and (d), the proof of the inclusion relation $\mathcal{J}(g_\lambda) \subseteq \mathrm{Esc}(g_\lambda)$ similarly follows on the lines of proof of Theorem 5.4.1.

To prove that $\mathrm{Esc}(g_\lambda) \subseteq \mathcal{J}(g_\lambda)$, we note that if $z_0 \in \mathrm{Esc}(g_\lambda)$ then, $z_0 \notin \mathsf{U}(x^*)$ or $z_0 \notin \mathsf{U}(-x^*)$, where $\mathsf{U}(x^*)$ (or $\mathsf{U}(-x^*)$) is the parabolic domain corresponding to the rationally indifferent real fixed point x^* (or $-x^*$) for $g_{\lambda_g^*}(z)$ (or $g_{-\lambda_g^*}(z)$). Consequently, by Theorem 6.3.3, $z_0 \in \mathcal{J}(g_\lambda)$ and $\mathrm{Esc}(g_\lambda) \subseteq \mathcal{J}(g_\lambda)$ follows. □

6.3.III Dynamics of $g_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| > \lambda_g^*$

The dynamics of $g_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| > \lambda_g^*$ is studied in the present subsection. In this case, we mainly prove that the Julia sets of $g_\lambda(z)$ explodes and equals the whole of complex plane.

Theorem 6.3.5. *Let* $g_\lambda \in \mathcal{S}$ *and* $|\lambda| > \lambda_g^*$. *Then, the Julia set of* $g_\lambda(z) = \mathbb{C}^\infty$ *for* $|\lambda| > \lambda_g^*$.

Proof. By Proposition 6.1.1, $g_\lambda(z)$ has only real critical values. By Proposition 6.1.2, $g_\lambda(z)$ has only one finite asymptotic value, namely, 0. If $\lambda > \lambda_g^*$ then, by Proposition 6.3.2(c), $g_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R}$ as $n \to \infty$. Therefore, the orbits of all singular values tend to ∞ under iteration of g_λ when $\lambda > \lambda_g^*$. Similarly, if $\lambda < -\lambda_g^*$, by Proposition 6.3.2(f),

$g_\lambda^n(x) \to -\infty$ for all $x \in \mathbb{R}$ as $n \to \infty$ and hence for $\lambda < -\lambda_g^*$, the orbits of all singular values tend to $-\infty$ under iteration of g_λ. Thus, the forward orbits of all singular values of $g_\lambda(z)$, for $|\lambda| > \lambda_g^*$ tend to infinity under iteration. Now, we only need to prove that the Fatou set of $g_\lambda(z)$ is empty if the orbits of all the singular values of $g_\lambda(z)$ tend to ∞ under iteration of $g_\lambda(z)$.

A component U of $\mathcal{F}(g_\lambda)$ can be (i) preperiodic, (ii) periodic or (iii) a wandering domain. We show that none of the possibilities (i), (ii) or (iii) could occur if $|\lambda| > \lambda_g^*$ and consequently it would follow that $\mathcal{F}(g_\lambda) = \emptyset$.

Let $\mathcal{F}(g_\lambda)$ contains either a preperiodic or a periodic component U. Since, if U is preperiodic, $f^m(U)$ is periodic for some non-negative integer m. Therefore, without loss of generality, we assume that U is a periodic component. By Theorem 1.1.7, only the following possibilities for U could occur: (a) U is a basin of attraction, (b) U is a parabolic domain, (c) U is a Siegel disk and (d) U is a Baker domain. However, for $|\lambda| > \lambda_g^*$, we show that none of the above cases (a), (b), (c) or (d) are possible.

Case (a) and (b): If the Fatou set of $g_\lambda(z)$ contains either a basin of attraction or a parabolic domain U then, by Theorem 1.1.8, U must contain atleast one critical point w (say). Since the orbits of critical points tend to ∞ under iteration, there exist a neighborhood N of the critical point $w \in U$ such that $\{g_\lambda^n(z)\}_{z \in N}$ tends to ∞ under iteration. But, as in the proof of Theorem 6.3.1, the absolute value of all the singular values of $g_\lambda(z)$ are less than are equal to $|\lambda|$ and therefore, by [40], there does not exist any component V of $\mathcal{F}(g_\lambda)$ such that $g_\lambda^n|_V \to \infty$ as $n \to \infty$, giving a contradiction. Thus, the cases (a) and (b) are not possible.

Case (c): If U is a Siegel disk of $g_\lambda(z)$, by Theorem 1.1.9, its boundary ∂U is contained in the forward orbits of the singular values of $g_\lambda(z)$. Since as in the case of critically finite entire functions [31], the forward orbits of all singular values of $g_\lambda(z)$ tend to ∞ under iteration of g_λ, it is impossible that the Sigel disks exist in the Fatou set of $g_\lambda(z)$.

A Shorter proof of Proposition 4.2.2, page 109.

Proof. The function $f_\lambda'(z) = \lambda\frac{e^z(z-1)+1}{z^2} = 0$ for $z \in H^+$ if and only if $e^z(z-1)+1=0$ for $z \in H^+$ if and only if $e^{-z} + (z-1) = 0$ for $z \in H^+$.

Supoose $\Re(z) > 0$ and $f_\lambda'(z) = 0$; then $e^{-z} + (z-1) = 0$, so that

$$\frac{\cos y - i\sin y}{e^x} = (1-x) - iy\,.$$

Suppose $y \neq 0$, then

$$\frac{\sin y}{y} = e^x > 1\,.$$

This is false for $y > 0$, and for $y < 0$ because $(\sin y)/y$ is an even function. We deduce that $y = 0$ and hence that $z = x > 0$ and

$$e^x = \frac{1}{1-x}\,.$$

This is false for $0 < x < 1$ (consider the two Taylor series); it is obviously false when $x = 1$, and also when $x > 1$ because then the left hand side is positive and the right hand side is negative. Therefore, $f_\lambda'(z)$ has no zeros in H^+.

CASE (d): Let U be a Baker domain. Then, $g_\lambda^n(z) \to \infty$ as $n \to \infty$ for all $z \in U$. But, as in the proof of Theorem 6.3.1 the set of all singular values of $g_\lambda(z)$ is bounded and therefore, by [40], there does not exist a component V of $\mathcal{F}(g_\lambda)$ such that $g_\lambda^n|_V \to \infty$ as $n \to \infty$, giving a contradiction. Therefore, $\mathcal{F}(g_\lambda)$ has no Baker domains.

Next, let U be a wandering domain. Let V be a component of the wandering domain and $w \in V$. Since $\{g_\lambda^n(w)\}$ is normal at w, there exist a neighborhood $N(w)$ such that $\{g_\lambda^n(z)\}$ is normal in $N(w)$. By Theorem 1.1.10, the finite limit function of $\{g_\lambda^n|_{N(w)}\}$ is contained in the derived set of forward orbits of singular values of $g_\lambda(z)$. Since the forward orbits of singular values tend to ∞ under iteration, the limit function of $\{g_\lambda^n|_{N(w)}\}$ must be ∞. This is a contradiction to the fact that the limit function of $\{g_\lambda^n|_{N(w)}\}$ is finite [40]. Therefore, the Fatou set of $g_\lambda(z)$ has no wandering domain.

Thus, for $|\lambda| > \lambda_g^*$, the occurrence of a component of the types (i), (ii) and (iii) in $\mathcal{F}(g_\lambda)$ is impossible. Therefore, $\mathcal{F}(g_\lambda) = \emptyset$ or equivalently, $\mathcal{J}(g_\lambda) = \mathbb{C}_\infty$ for $|\lambda| > \lambda_g^*$. □

Remark 6.3.5. *Theorems 6.3.1 and 6.3.3 show that the Fatou sets of $g_\lambda(z)$ for $|\lambda| \le \lambda_g^*$ is non-empty and unbounded (c.f. Remarks 6.3.1(i) and 6.3.3.(i)). Theorem 6.3.5 shows that when the parameter $|\lambda|$ crosses the value λ_g^*, the Julia set suddenly explodes and equals the extended complex plane, making the Fatou sets empty. It seems that the functions in the class $\mathcal{G}$ are the first examples of non-critically finite entire functions having such chaotic bursts in their dynamics.*

The following is a characterization of the Julia set of $g_\lambda(z)$ for $|\lambda| > \lambda_g^*$ analogous to those of Theorems 6.3.2 and 6.3.4:

Theorem 6.3.6. *Let $g_\lambda \in \mathcal{S}$ and $Esc(g_\lambda) = clo\ \{z \in \mathbb{C} : g_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$ be the closure of the set of escaping points of $g_\lambda(z)$. If $|\lambda| > \lambda_g^*$, then the Julia set $\mathcal{J}(g_\lambda) = Esc(g_\lambda)$.*

Proof. The proof of the inclusion relation $\mathcal{J}(g_\lambda) \subseteq \mathrm{Esc}(g_\lambda)$ follows on the lines similar

to that of Theorem 6.3.4. The reverse inclusion $\mathrm{Esc}(g_\lambda) \subseteq \mathcal{J}(g_\lambda)$ obviously follows from Theorem 6.3.5.

□

6.4 Examples

In the present section, certain interesting examples of the family $\mathcal{S}$, viz., (i) $\mathcal{I} = \{\lambda I_0(z) : \lambda \in \mathrm{R} \setminus \{0\}\}$, where $I_0 \in \mathcal{G}$ is the well known modified Bessel function of zero order arising as the denominator of the separately convergent modified general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{-z^2/(2n)^2}{1 + z^2/(2n)^2} \right)$ and (ii) $\mathcal{M}_k = \{\lambda G_{2k}(z) : G_{2k}(z) = F_{2k}(iz)/F_{2k}(0),\ \lambda \in \mathrm{R}\setminus\{0\}\}$, where $G_{2k} \in \mathcal{G}$ with fixed $k = 1, 2, \ldots$ and $F_{2k}(z) = \int_0^\infty e^{-t^{2k}} \cos zt\, dt$ are given. Further, as applications of Theorems 6.3.2, 6.3.4 and 6.3.6, the picture of the Julia sets of functions in the family $\mathcal{I}$ is computationally generated for various values of λ.

6.4.I Dynamics of $G_{2k}(z) = \frac{F_{2k}(iz)}{F_{2k}(0)}$, where $k > 1$ is a fixed integer and $F_{2k}(z) = \int_0^\infty e^{-t^{2k}} \cos zt\, dt$

For each $k > 1$, the integral $\int_0^\infty e^{-t^{2k}} \cos zt\, dt$ converges uniformly for all $z \in \mathrm{C}$. Therefore, $F_{2k}(z)$ is an entire function. The functions $F_{2k}(z)$ has Taylors series expansion at $z = 0$ is given by ([100], p271):

$$F_{2k}(z) = \frac{1}{2k} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{2n+1}{2k}\right)}{\Gamma(2n+1)} z^{2n}$$

where, $\Gamma(z)$ is the gamma function. In the followin proposition, for each $k > 1$ the function $G_{2k} \in \mathcal{G}$ is proved:

Proposition 6.4.1. *Let $G_{2k}(z) = F_{2k}(iz)/F_{2k}(0)$ for a fixed integer $k > 1$ and $F_{2k}(z) = \int_0^\infty e^{-t^{2k}} \cos zt\, dt$. Then, $G_{2k} \in \mathcal{G}$ for a fixed integer $k > 1$.*

Proof. We prove that $\frac{F_{2k}(i\sqrt{z})}{F_{2k}(0)}$ belongs to the class $\mathcal{F}$ and consequently the function $G_{2k}(z) = F_{2k}(iz)/F_{2k}(0)$ belongs to the class $\mathcal{G}$ would follow.

Since $F_{2k}(i\sqrt{z}) = \frac{1}{2k}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{2n+1}{2k})}{\Gamma(2n+1)}z^n$, by (1.3.1), it follows that the order ρ of $F_{2k}(i\sqrt{z})$ is found to be

$$\rho = \limsup_{n\to\infty}\frac{n\log n}{\log\Gamma(2n+1)-\log\Gamma(\frac{2n+1}{2k})} = \limsup_{n\to\infty}\frac{n\log n}{2n\log 2n - \frac{2n}{2k}\log n + \mathcal{O}(n)} = \frac{k}{2k-1}.$$

Clearly, $k > 1$ implies that $\frac{1}{2} < \rho < 1$. Therefore, the function $\frac{F_{2k}(i\sqrt{z})}{F_{2k}(0)}$ satisfies the condition (6.1.1*(i)*). Since the function $F_{2k}(i\sqrt{z})$ has only negative real zeros ([100], p271), the function $\frac{F_{2k}(i\sqrt{z})}{F_{2k}(0)}$ satisfies the condition (6.1.1*(ii)*). Further, for $x > 0$, $F_{2k}(i\sqrt{-x}) = \int_0^\infty e^{-t^{2k}}\cos(i\sqrt{-x}t)dt = \int_0^\infty e^{-t^{2k}}\cos(-\sqrt{x}t)dt$, it is easily seen that the function $\frac{F_{2k}(i\sqrt{z})}{F_{2k}(0)}$ satisfies the conditions (6.1.1*(iii)*) and (6.1.1*(iv)*). Thus, for any integer $k > 1$, the function $\frac{F_{2k}(i\sqrt{z})}{F_{2k}(0)}$ belongs to the class $\mathcal{F}$. □

Let

$$\mathcal{M}_k = \{\lambda G_{2k}(z)\ :\ G_{2k}(z) = \frac{F_{2k}(iz)}{F_{2k}(0)},\ \lambda \in \mathrm{R}\setminus\{0\}\}$$

where $G_{2k} \in \mathcal{G}$ and $k > 1$ is a fixed integer. Let

$$\lambda_G^* = \frac{1}{G'_{2k}(\,x^*\,)} \tag{6.4.1}$$

where, x^* is the unique positive real root of the equation $\phi(x) = G_{2k}(x) - xG'_{2k}(x)$. Theorem 6.2.2 gives that bifurcation in the dynamics of functions in the family $\mathcal{M}_k$ occurs at $|\lambda| = \lambda_G^*$

If $0 < |\lambda| < \lambda_G^*$, by Theorem 6.3.1, the Fatou set of $\lambda G_{2k}(z)$ is the basin of attraction of the attracting real fixed point. If $|\lambda| = \lambda_G^*$, by Theorem 6.3.3, the Fatou set of $\lambda G_{2k}(z)$ is the parabolic domain corresponding to the rationally indifferent real fixed point. Theorem 6.3.5 gives that the Julia set of $\lambda G_{2k}(z)$ explodes and equals the extended complex plane for $|\lambda| > \lambda_G^*$ and consequently, the Fatou set of $\lambda G_{2k}(z)$ is empty set. Thus, the chaotic burst occurs in the dynamics of functions in the family $\mathcal{M}_k$.

6.4.II Dynamics of Modified Bessel function $I_0(z)$

Let

$$D(z) \equiv \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{z^2/(2n)^2}{1 - z^2/(2n)^2} \right) \tag{6.4.2}$$

be a modified general T-fraction and $B_n(z)$ denote the denominator of the nth approximant of the continued fraction (6.4.2). Since $\sum |\frac{1}{(2n)^2}| < \infty$, by Theorem 2.5.1, it follows that the sequence $\{B_n(z)\}_{n=1}^{\infty}$ converges to the entire function $B(z) = 1 + \sum_{k=1}^{\infty} q_k z^{2k}$. It is readily seen from (1.2.2) that the function $B_n(z)$ satisfies the following recurrence relation

$$B_n(z) = B_{n-1}(z) + \frac{z^2}{(2n)^2} [B_{n-1}(z) - B_{n-2}(z)]$$

with initial conditions $B_{-1} \equiv 0$ and $B_0 \equiv 1$. Therefore, it is easily seen that the function $B_n(z)$ is given by $B_n(z) = 1 + \sum_{k=1}^{n} \frac{z^{2k}}{F_1 F_2 \cdots F_k}$ with $F_k = \frac{1}{(2k)^2}$, $k = 1, 2, \ldots$. Now,

$$\lim_{n\to\infty} B_n(z) = \lim_{n\to\infty} \left(1 + \sum_{k=1}^{n} \frac{z^{2k}}{F_1 F_2 \cdots F_k}\right) = 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{((2)^k (k!))^2} = I_0(z) ,$$

where, $I_0(z)$, called the modified Bessel function of zero order [106]. Consequently, $B(z) = I_0(z)$. Thus, the function $I_0(z)$ is represented as the limit function of the sequence of ominators of the approximants of the continued fraction (6.4.2).

Let $J_0(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\, z^{2k}}{((2)^k (k!))^2}$, be the Bessel function (of first kind) of zero order [106]. Since, [106], $I_0(z) = J_0(iz)$ for all $z \in$ C and $J_0(z)$ has only infinitely many real zeros, the function $I_0(\sqrt{z})$ has only infinitely many negative real zeros in the complex plane. Therefore, $I_0(\sqrt{z})$ satifies (6.1.1*(ii)*).

For $x \geq 0$, $I_0(\sqrt{-x}) = I_0(i\sqrt{x}) = J_0(-\sqrt{x})$. Further, since, [106], $|J_0(x)| \leq 1$ for all $x \in$ R and $J_0(x) \to 0$ as $|x| \to \infty$, it follows that $|I_0(\sqrt{-x})| = |J_0(-\sqrt{x})| \leq 1 = I_0(0)$ for all $x > 0$ and $I_0(\sqrt{-x}) \to 0$ as $x \to \infty$. Consequently, the function $I_0(\sqrt{z})$ satifies the conditions (6.1.1*(iii)*) and (6.1.1*(iv)*). Thus, the function $I_0(\sqrt{z})$ belongs to the class $\mathcal{F}$. $\square$

Let

$$\mathcal{I} \equiv \{\lambda I_0(z) : \lambda \in \mathrm{R} \setminus \{0\}\}$$

be one parameter family of non-critically finite entire functions. In view of the results found in Sections 6.2 and 6.3, the dynamics of functions in the family $\mathcal{I}$ is easily described as follows:

Let

$$\lambda_I^* = \frac{1}{I_0'(\,x^*\,)} \tag{6.4.3}$$

where, x^* is the unique positive real root of the equation $\phi(x) = I_0(x) - xI_0'(x)$. The numerical computation of the root x^* of the equation $\phi(x) = 0$ by the bisection method gives $x^* \approx 1.6082$. Therefore, $\lambda_I^* \approx 0.91431$. Theorem 6.2.2 gives that bifurcation in the dynamics of functions in the family $\mathcal{I}$ occurs at $|\lambda| = \lambda_I^* \approx 0.91431$

If $0 < |\lambda| < \lambda_I^*$, by Theorem 6.3.1, the Fatou set of $\lambda I_0(z)$ is the basin of attraction of the attracting real fixed point. If $|\lambda| = \lambda_I^*$, by Theorem 6.3.3, the Fatou set of $\lambda I_0(z)$ is the parabolic domain corresponding to the rationally indifferent real fixed point. Theorem 6.3.5 gives that the Julia set of $\lambda I_0(z)$ explodes and equals the extended complex plane for

$|\lambda| > \lambda_I^*$ and consequently, the Fatou set of $\lambda I_0(z)$ is empty set. Thus, the chaotic burst occurs in the dynamics of functions in the family $\mathcal{I}$ at the parameter value $\lambda_I^* \approx 0.91431$.

Since $I_0 \in \mathcal{G}$, Theorems 6.3.2, 6.3.4 and 6.3.6 are applicable and consequently the algorithm of Section 4.6 could be used to generate the pictures of Julia sets of $\lambda I_0(z)$ for various values of λ. Thus, let $R = \{z \in \mathbb{C} : -5 \leq \Re(z) \leq 5 \text{ and } -5 \leq \Im(z) \leq 5\}$, the maximum number of iterations $N = 200$ and $M = 100$. The resulting pictures of the Julia set of $\lambda I_0(z)$ for $\lambda = 0.9 < \lambda_I^*$, $\lambda = 0.91431 \approx \lambda_I^*$ and $\lambda = 0.92 > \lambda_I^*$ are shown in Figure 6.1.

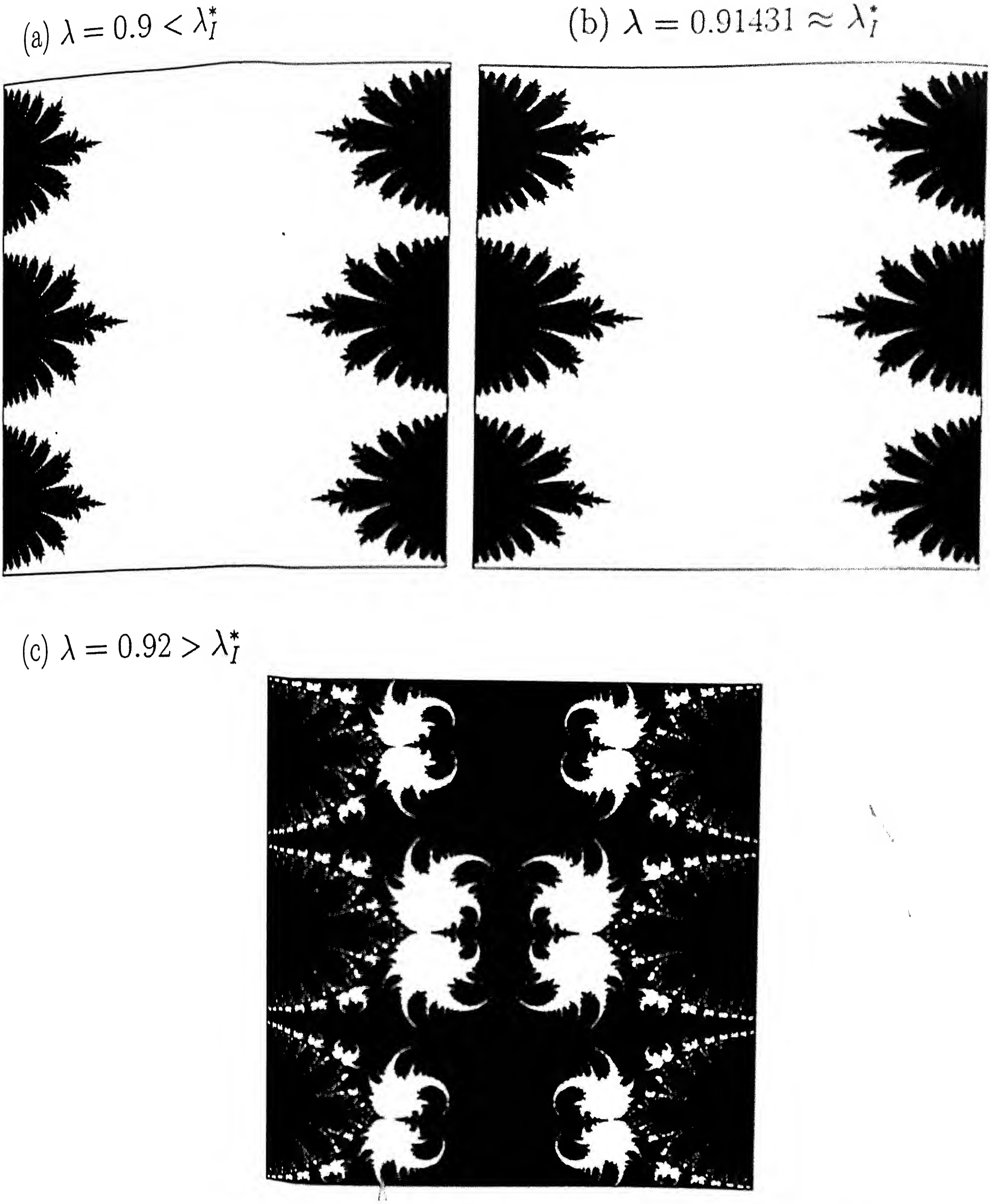


Figure 6.1: Julia sets of $\lambda I_0(z)$ for $\lambda = 0.9 < \lambda_I^*$, (b) $\lambda = 0.91431 \approx \lambda_I^*$ and (c) $\lambda = 0.92 > \lambda_I^*$.

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